Segmentation using Level Set Methods

Region Based Active Contours

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Active Contours (Snakes)

- Active contour model (snakes) can be written as:

$$E_{AC}(\mathcal{C}) = \int_0^1 \alpha|\mathcal{C}'(q)|^2 + \beta|\mathcal{C}''(q)|^2 dq + \int_0^1 P(\mathcal{C}(q))dq,$$

  where P is the potential field.
- The deformable contour (snake) is a mapping:

$$\mathcal{C}(q) : [0, 1] \to \mathbb{R}^2, \quad q \mapsto \mathcal{C}(q) = (x(q), y(q))^T.$$

- Potential P can be:
  - Edges: $P_{edge}(\mathcal{C}(q)) = -|\nabla(G_\sigma * f(\mathcal{C}(q)))|^2$
  - Lines (high intensity): $P_{line}(\mathcal{C}(q)) = -G_\sigma * f(\mathcal{C}(q))$
  - Combination: $P(\mathcal{C}(q)) = -w_{line}P_{line} - w_{edge}P_{edge}$
  - Often $P(\mathcal{C}(q)) = g(|\nabla G_\sigma * I(\mathcal{C}(q))|)$ where $g : [0, \infty) \to \mathbb{R}^+$ is a strictly decreasing function: $g(r) \to 0$ as $r \to \infty$.

Active Contours: Evolution equation

- The snake evolution equation:

$$\frac{\partial \mathcal{C}}{\partial t} = \alpha \mathcal{C}''(s) - \beta \mathcal{C}''''(s) - k_1 \mathbf{n}(s) - k_2 \frac{\nabla P(\mathcal{C}(s))}{|\nabla P(\mathcal{C}(s))|}$$

  where

$$P(\mathcal{C}(s)) = -w_{line}(G_\sigma * f(\mathcal{C}(s))) + w_{edge}|\nabla(G_\sigma * f(\mathcal{C}(s)))|^2$$

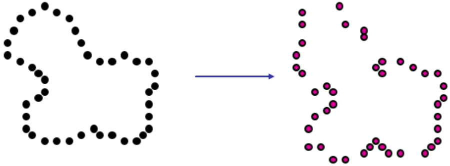
- In matrix form:

$$(I + \tau A)X^{t+1} = X^t + \tau F(X^t)$$

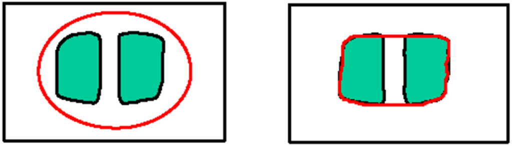
- Parameter choice (better but not obligatory):
  - α controls elasticity
  - β controls stiffness
  - k_1 sign controls inflate or deflate
  - $|\tau k_1| < |\tau k_2| < 1$
  - τ controls the snake speed

Problems with Parametric Curves

- Reparametrization needed (hard with surfaces in 3D)



- Cannot handle topological changes



- Hard to extend to 3D

$$\gamma\frac{\partial \mathbf{v}}{\partial t} - \frac{\partial}{\partial s}\left(w_{10}\frac{\partial \mathbf{v}}{\partial s}\right) - \frac{\partial}{\partial r}\left(w_{01}\frac{\partial \mathbf{v}}{\partial r}\right) + 2\frac{\partial^2}{\partial s \partial r}\left(w_{11}\frac{\partial^2 \mathbf{v}}{\partial s \partial r}\right)$$
$$+\frac{\partial^2}{\partial s^2}\left(w_{20}\frac{\partial^2 \mathbf{v}}{\partial s^2}\right) + \frac{\partial^2}{\partial r^2}\left(w_{02}\frac{\partial^2 \mathbf{v}}{\partial r^2}\right) + \nabla P(\mathbf{v}(s,r)) = 0,$$

Geodesic Active Contours

- Caselles et.al. 1995, Kichenassamy et al. 1995
- Curve with minimal geodesic length is searched.

$$E_{GAC}(\mathcal{C}) = \int_0^{L(\mathcal{C})} g(|\nabla G_\sigma * I(\mathcal{C}(s))|)ds =$$
$$= \int_0^1 g(|\nabla G_\sigma * I(\mathcal{C}(q))|)|\mathcal{C}'(q)|dq$$

where $g : [0, \infty) \to \mathbb{R}^+$ is a strictly decreasing function such that $g(r) \to 0$ as $r \to \infty$, e.g.,

$$g(|\nabla G_\sigma * I(x, y)|) = \frac{1}{1 + |\nabla G_\sigma * I(x, y)|}$$

- The curve is attracted by image edges, where the weight $g(|\nabla G_\sigma * I(C(q))|)$ is small.
- Energy functional is not convex and therefore there are several local minima.

Geodesic Active Contours: Evolution Equation

- Evolution equation (i.e. Euler-Lagrange equation with $\frac{\partial \mathcal{C}}{\partial t}$ on the left side) for geodesic active contours is

$$\frac{\partial \mathcal{C}}{\partial t} = \left(g\kappa - (\nabla g \cdot \mathbf{n})\right)\mathbf{n}$$

  where **n** is curve normal (vector) and κ is curvature (scalar)
- This equation can be rewritten in level-set framework.
- The curve is embeded in SDF u (with evolution speed $\nu = g\kappa - \nabla g \cdot n$).
  The evolution equation becomes

$$\frac{\partial u}{\partial t} = g\kappa|\nabla u| - \nabla g \cdot \nabla u$$

Level Sets Idea

- Curve $\mathcal{C}$ is represented implicitly as a zero level set of a higher-dimensional function $u : \mathbb{R}^2 \to \mathbb{R}$.

  $$\mathcal{C} = \{(x, y) : u(x, y) = 0\}$$

  In level set formulation, the curve evolution according to

  $$\frac{\partial \mathcal{C}}{\partial t} = \beta \boldsymbol{n}$$

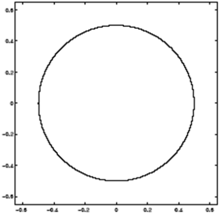
  leads to the evolution of embedding function u according to

  $$\frac{\partial u}{\partial t} + \beta |\nabla u| = 0$$

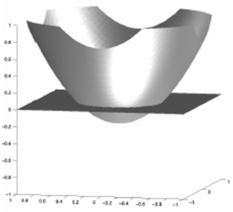
  where $\boldsymbol{n}$ is curve normal and β is the evolution speed (scalar).

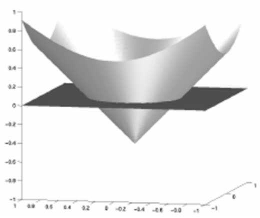
Level Sets Idea

- Curve



- Different embedding functions $u(x, y)$

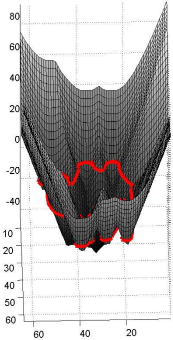




Signed Distance Function (SDF): Example

- For $C = \partial\Omega$, SDF d defined by:
  $$d(\mathbf{x}) = \begin{cases} -\min_{\mathbf{y}\in C} |\mathbf{x}-\mathbf{y}| & \text{if } \mathbf{x} \in \Omega^- \\ +\min_{\mathbf{y}\in C} |\mathbf{x}-\mathbf{y}| & \text{if } \mathbf{x} \in \Omega^+ \cup C \end{cases}$$
- Normal $n = \nabla d$, $|\nabla d| = 1$
- Curvature $\kappa = \nabla \cdot \nabla d = \nabla^2 d = \Delta d$

digital shape



SDF $d(\mathbf{x})$

Geodesic Active Contours: Evolution Equation

- The evolution equation

$$\frac{\partial u}{\partial t} = g\kappa|\nabla u| - \nabla g \cdot \nabla u$$

  contains curvature motion and velocity field motion.
- It is solved using level set methods
- We can add normal direction motion (baloon force)

$$\frac{\partial u}{\partial t} = (c + \kappa)g|\nabla u| - \nabla g \cdot \nabla u$$

Geodesic Active Contours: Iterative Scheme

- The equation

$$u_t = (c + \epsilon\kappa)g|\nabla u| + \beta\nabla g \cdot \nabla u$$

consists of three types of motion we have discussed before:

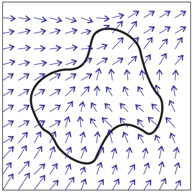
  - normal direction motion with speed $cg(|\nabla G_\sigma * I(x, y)|)$
  - curvature motion multiplied by a factor $\epsilon g(|\nabla G_\sigma * I(x, y)|)$
  - external velocity field motion given by $\beta\nabla g$.

- Therefore, we have

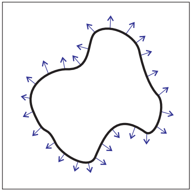
$$u_{ij}^{k+1} = u_{ij}^{k} + \tau \cdot [\text{Normal}(cg) + \text{Curvature}(\epsilon g) - \text{Velocity}(\beta\nabla g)].$$

Three types of motion

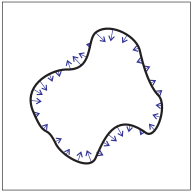
external velocity



normal motion



curvature motion



All Types of Motion Together

The general equation

$$u_t = a|\nabla u| - \epsilon\kappa|\nabla u| + \beta V \cdot \nabla u$$

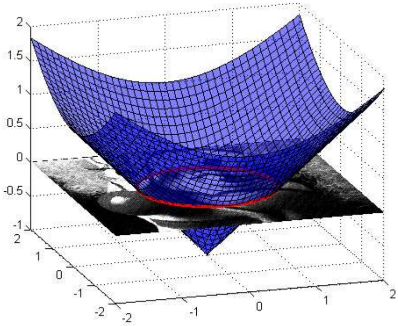
is discretized with the numerical scheme

$$u_{ij}^{n+1} = u_{ij}^{n} + \tau \left[\begin{array}{c} [\max(cg_{ij},0)\nabla^{+} + \min(cg_{ij},0)\nabla^{-}] + \\ \\ +[\epsilon g_{ij} K_{ij}^{n} \sqrt{(D_{ij}^{0x})^2 + (D_{ij}^{0y})^2}] + \\ \\ \left[\begin{array}{c} \max(w_{ij}^{n},0)D_{ij}^{-x} + \min(w_{ij}^{n},0)D_{ij}^{+x} \\ + \max(v_{ij}^{n},0)D_{ij}^{-y} + \min(v_{ij}^{n},0)D_{ij}^{+y} \end{array} \right] \end{array} \right],$$

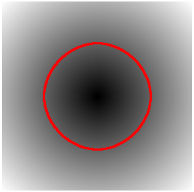
Note: $w_{ij} = g'_x(ih, jh)$ and $v_{ij} = g'_y(ih, jh)$

Stability condition: The most restrictive (curvature) term forces $\tau = \mathcal{O}(h^2)$

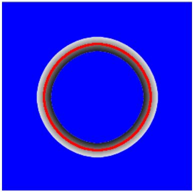
Level Set Methods: Narrow Band



Whole domain

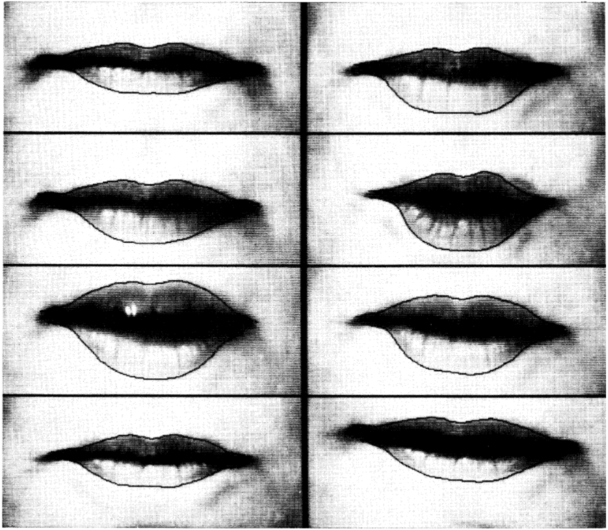


Narrow band



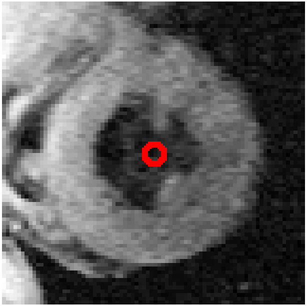
- We can compute the evolution in a narrow band around the zero level set only!

Active Contours: Example

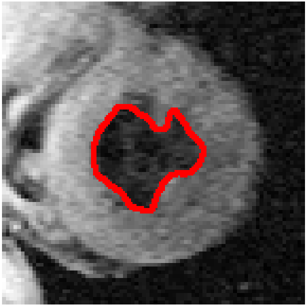


Source: [Kass et. al. 1987]

Geodesic Active Contours: Example



Initial contour

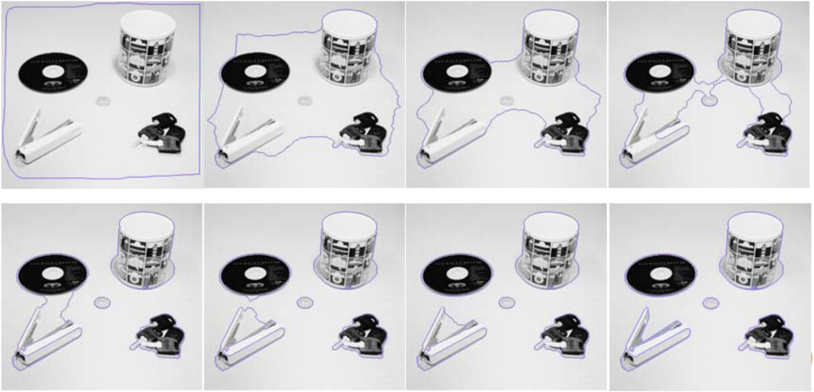


Result

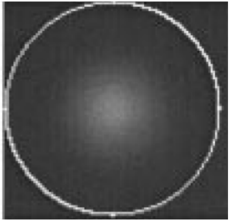
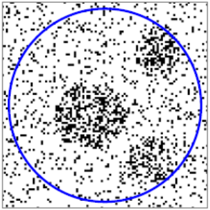
$\tau = 0.25, c = 1, \epsilon = 0.5,$

$\beta = 0.5, p = 2, \sigma = 2.0$

Geodesic Active Contours: Example



What if no / weak / noisy edges?

- Region interior was not considered in previously discussed active contours!
- No region homogeneity was required (image structures under the curve are important for solution only).

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Mumford-Shah Functional

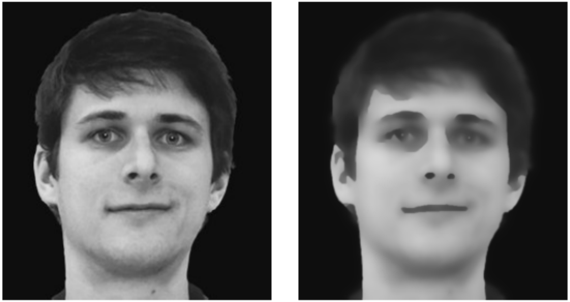
- Functional formulation of segmentation.
- Segmentation $(u, \mathcal{C})$ of an image $f : \Omega \to \mathbb{R}$ is defined as the minimizer of

$$E_{MS}(u, \mathcal{C}) = \lambda \int_{\Omega} (u - f)^2 dx + \beta \int_{\Omega \setminus \mathcal{C}} |\nabla u|^2 dx + \mu |\mathcal{C}|$$

  where $u(x)$ is smoothed version of $f(x)$, and $\mathcal{C}$ is an edge set curve where u is allowed to be discontinuous
- First term penalises deviations from original image f
- Second term penalises variations within each segment
- Third term penalises the edge length $|\mathcal{C}|$
- Mathematically very difficult:
  - We are looking for edges $\mathcal{C}$ and image u
  - Non-convex
  - Complicated and computationally expensive
- No unique solution in general.
- Several simplifications of functional E_{MS} were proposed

Mumford-Shah Functional: Example

f | Mumford–Shah piecewise-smooth approximation



From [P. Getreuer, Chan–Vese Segmentation, IPOL Journal, 2012]

Mumford-Shah Functional

- Piecewise constant formulation
- Segmentation $(u, \mathcal{C})$ of an image $f : \Omega \to \mathbb{R}$ is defined as the minimiser of

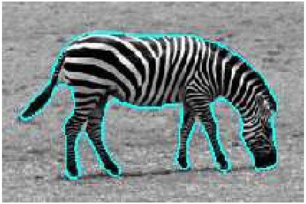
$$E_{MS1}(u, \mathcal{C}) = \int_{\Omega} (u - f)^2 dx + \mu |\mathcal{C}|$$

  where $u(x)$ u is required to be constant on each connected component of $\Omega \setminus \mathcal{C}$
- Existence of a solution proved by [Mumford and Shah, 1989] and [Morel and Solimini, 1994]

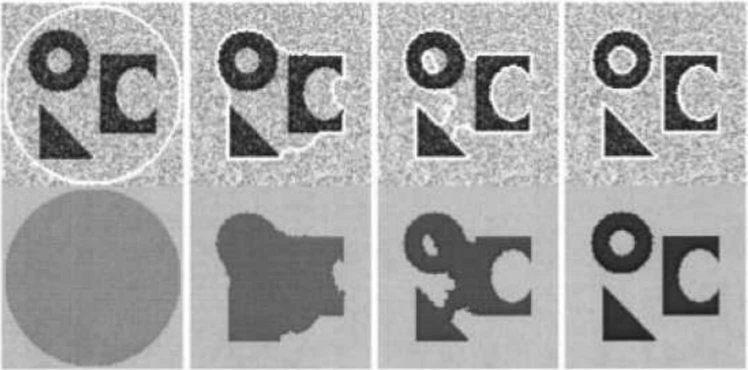
Contents

Active-Contours Without Edges

- Chan and Vese 1998
- **Given**: Image $f : \Omega \to \mathbb{R}$
- **Goal**: Segmentation of Ω into two regions (possibly disconnected)
- Curve evolution is based on region information (but not on edges)
- Can be extended to segment color and textured images.

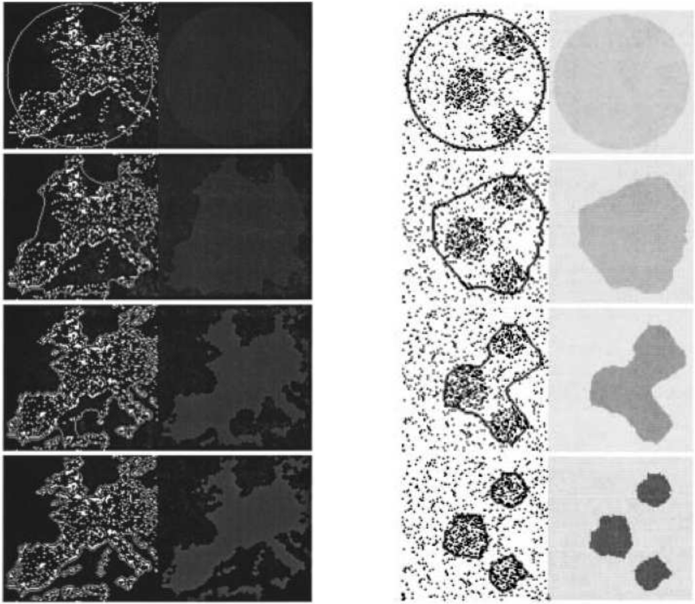


Active Contours Without Edges: Example 1



Source: Chan, Vese, Active Contours Without Edges, 2001

Active Contours Without Edges: Example 2



Source: Chan, Vese, Active Contours Without Edges, 2001

Active-Contours Without Edges: Idea

- Regions are separated by a curve $\mathcal{C}$.
- In each region, a constant grey-value is supposed to approximate the image.
- **Data term** penalises the deviation from the piecewise constant approximation of the input image
- **Regularity term** impose regularity constraints for the curve (requires curve of minimal length).

Active-Contours Without Edges: Functional

- Let $f(x) : \Omega \to \mathbb{R}$ is the input image.
- Let $\mathcal{C}$ be the boundary between two regions Ω_1 and Ω_2.
  $\Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{C}$
- Chan-Vese functional is defined

  $$E_{CV}(\mathcal{C}, c_1, c_2) = \mu L(\mathcal{C}) + \nu A(\Omega_1) + \\ + \lambda_1 \int_{\Omega_1} |f(x) - c_1|^2 dx + \lambda_2 \int_{\Omega_2} |f(x) - c_2|^2 dx$$

  where $\mu \geq 0, \nu \geq 0, \lambda_1, \lambda_2 \geq 0$ are given fixed parameters (weights), $L(\mathcal{C})$ denotes the length of $\mathcal{C}$, $A(\Omega_1)$ denotes the area inside $\mathcal{C}$, and c_1 and c_2 are mean intensity values of two distinct regions
- We minimize the functional with respect to c_1,c_2, and $\mathcal{C}$.

Equivalence with Mumford-Shah Functional

- Chan-Vese functional:

$$E_{CV}(\mathcal{C}, c_1, c_2) = \mu L(\mathcal{C}) + \nu A(\Omega_1) + \\ + \lambda_1 \int_{\Omega_1} |f(x) - c_1|^2 dx + \lambda_2 \int_{\Omega_2} |f(x) - c_2|^2 dx$$

- Mumford-Shah functional:

$$E_{MS}(u, \mathcal{C}) := \lambda \int_{\Omega} (u - f)^2 dx + \beta \int_{\Omega \setminus K} |\nabla u|^2 dx + \mu |\mathcal{C}|$$

- Key difference:
  - Only two regions (in MS $\lambda = \lambda_1 = \lambda_2$)
  - Area term
  - Piecewise constant approximation u with 2 values

Chan-Vese Functional: Theory

- For given $\mathcal{C}$, the optimal values c_1 and c_2 are uniquely determined as the average gray values of f within their respective regions.
- There exist a curve $\mathcal{C}$ of finite length that minimises $E_{CV}(\mathcal{C}) = E_{CV}(\mathcal{C}, c_1(\mathcal{C}), c_2(\mathcal{C}))$ (existence of solution).
- Energy is non-convex, therefore there can exist more then one local minimum.

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Active-Contours Without Edges: Implementation

- Level set formulation: The curve $\mathcal{C}$ is represented as a zero level set of a continuous function $u : \Omega \to \mathbb{R}$. We get

$$
\begin{aligned}
E_{CV}(\mathcal{C}, c_1, c_2) &= E_{CV}(u, c_1, c_2) = \\
&= \mu \int_\Omega |\nabla H(u(x))| dx + \nu \int_\Omega H(u(x)) dx + \\
&+ \lambda_1 \int_\Omega (f(x) - c_1)^2 H(u(x)) dx + \\
&+ \lambda_2 \int_\Omega (f(x) - c_2)^2 (1 - H(u(x))) dx
\end{aligned}
$$

where $H(u)$ is Heaviside function:

$$
H(u) = \begin{cases} 1 & \text{for } u \geq 0 \\ 0 & \text{for } u < 0 \end{cases}
$$

Active-Contours Without Edges: Implementation

- $H(u)$ is Heaviside function:

$$H(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

- $\delta(t) = \frac{d}{dt}H(t)$ is Dirac delta function:

$$\int_{-\infty}^{+\infty} \delta(x)f(x)dx = f(0), \quad \int_{-\infty}^{+\infty} \delta(x)dx = 1$$

- Curve length:

$$L(\mathcal{C}) = \int_\Omega |\nabla H(u(x))|dx = \int_\Omega \delta(u(x))|\nabla u(x)|dx$$

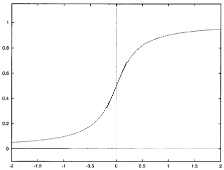
- Constants are average gray values of regions:

$$c_1 = \frac{\int_\Omega f(x)H(u(x))dx}{\int_\Omega H(u(x))dx}, \quad c_2 = \frac{\int_\Omega f(x)(1 - H(u(x)))dx}{\int_\Omega (1 - H(u(x)))dx}$$

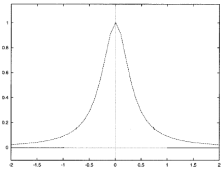
Active-Contours Without Edges: Implementation

- Regularized version of H is used:

$$H_\epsilon(u) = \frac{1}{2}(1 + \frac{2}{\pi} tan^{-1}\left(\frac{u}{\epsilon}\right)).$$



- The corresponding δ_ϵ:



Evolution Equation

- Minimization of CV functional leads to the following evolution equation:

$$\frac{\partial u}{\partial t} = \delta_\epsilon(u)\left[\mu\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right) - \nu - \lambda_1(f-c_1)^2 + \lambda_2(f-c_2)^2\right]$$

where

$$c_1 = \frac{\int_\Omega f(x)H(u(x))dx}{\int_\Omega H(u(x))dx}, \quad c_2 = \frac{\int_\Omega f(x)(1-H(u(x)))dx}{\int_\Omega(1-H(u(x)))dx}$$

and δ_ϵ is regularized Dirac function. It is a derivative of

$$H_\epsilon(u) = \frac{1}{2}(1+\frac{2}{\pi}tan^{-1}\left(\frac{u}{\epsilon}\right)).$$

- Notice, the first term is the curvature κ
- u is changed only within narrow band where $\delta_\epsilon \neq 0$.

Discrete Evolution Equation

The discrete evolution equation is:

$$u_{ij}^{k+1} = u_{ij}^{k} + \tau \delta_{\epsilon}(u_{ij}^{k}) \cdot \left[\text{Curvature}(\mu) - \lambda_1 (f_{ij} - c_1)^2 + \lambda_2 (f_{ij} - c_2)^2 \right]$$

where c_1 and c_2 are average intensities of foreground and background at time $k\tau$ and Curvature(μ) is the curvature term applied at u_{ij} (see previous Lecture).

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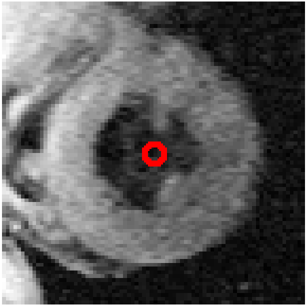
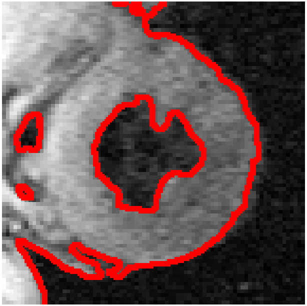
Example: Comparison to GAC

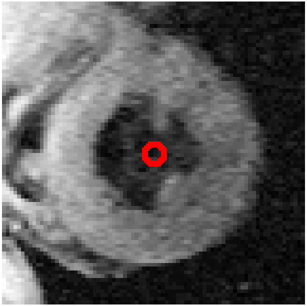


Initial contour

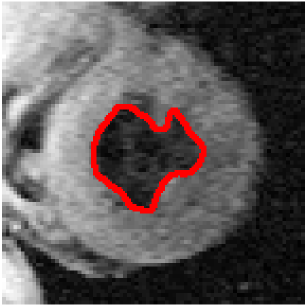


Result Chan-Vese
$\tau = 0.5, \lambda_1 = \lambda_2 = 0.05,$
$\epsilon = 10, \mu = 10, \nu = 0$

Example: Comparison to GAC



Initial contour



Result GAC
$\tau = 0.25, c = 1, \epsilon = 0.5,$
$\beta = 0.5, p = 2, \sigma = 2.0$

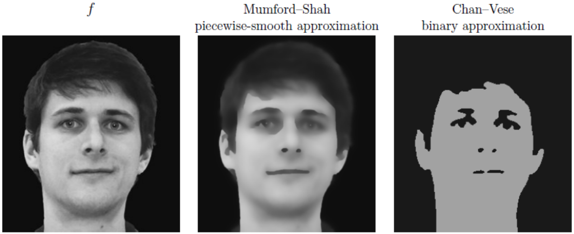
Example: Comparison to Mumford-Shah

- Chan-Vese functional:

$$E_{CV}(\mathcal{C}, c_1, c_2) = \mu L(\mathcal{C}) + \nu A(\Omega_1) + \\ + \lambda_1 \int_{\Omega_1} |f(x) - c_1|^2 dx + \lambda_2 \int_{\Omega_2} |f(x) - c_2|^2 dx$$

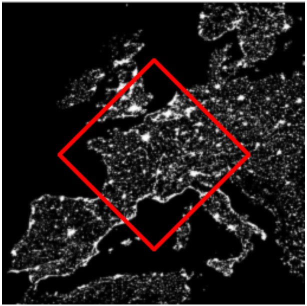
- Mumford-Shah functional:

$$E_{MS}(u, \mathcal{C}) := \lambda \int_{\Omega} (u - f)^2 dx + \beta \int_{\Omega \setminus K} |\nabla u|^2 dx + \mu |\mathcal{C}|$$

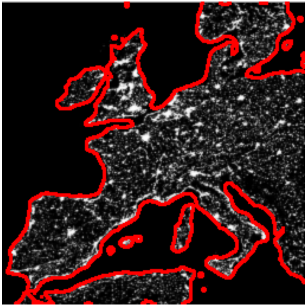


From [P. Getreuer, Chan–Vese Segmentation, IPOL Journal, 2012]

Example



Initial contour

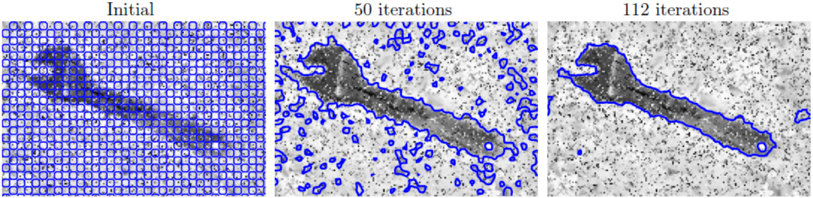


Result

$\tau = 0.5, \lambda_1 = 1, \lambda_2 = 0.02,$
$\epsilon = 10, \mu = 10, \nu = 0$

Example: Initial u_0

- u_0 does not have to be SDF
- $u_0(x, y) = \sin(\frac{\pi}{5}x)\sin(\frac{\pi}{5}y)$

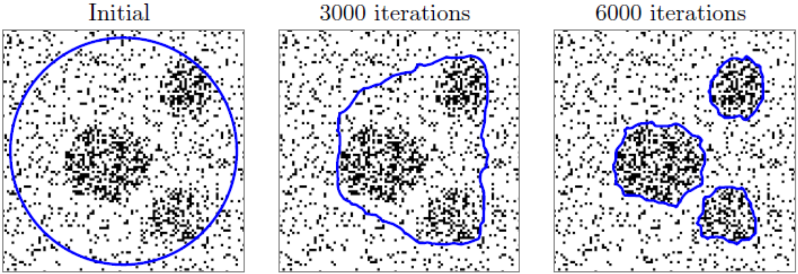


$$\tau = 0.5, \lambda_1 = 1, \lambda_2 = 1,$$
$$\mu = 0.2, \nu = 0, \epsilon = 1$$

From [P. Getreuer, Chan–Vese Segmentation, IPOL Journal, 2012]

Example: Initial u_0

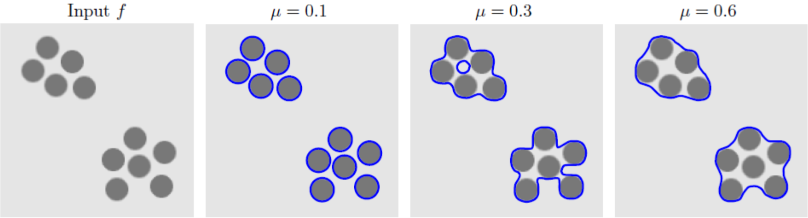
- u_0 is circle SDF



$$\tau = 0.5, \lambda_1 = 1, \lambda_2 = 1,$$
$$\mu = 0.3, \nu = 0, \epsilon = 1$$

From [P. Getreuer, Chan–Vese Segmentation, IPOL Journal, 2012]

Example: Dependence on μ

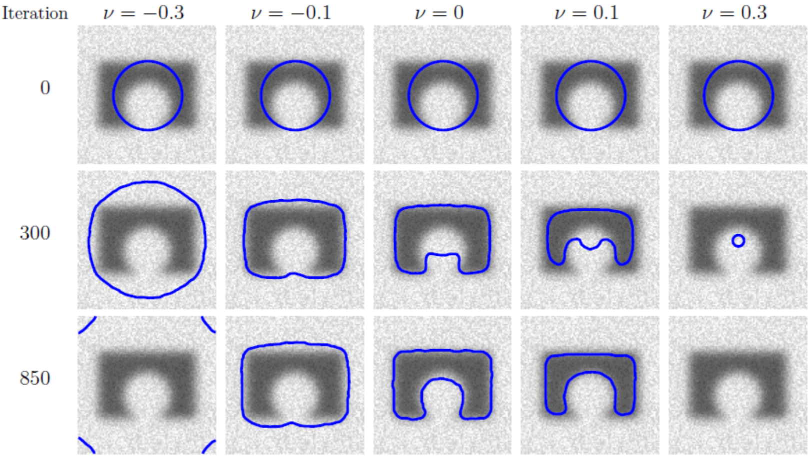


Chan–Vese results with different values of μ.

$$\tau = 0.5, \lambda_1 = 1, \lambda_2 = 1,$$
$$\nu = 0, \epsilon = 1$$

From [P. Getreuer, Chan–Vese Segmentation, IPOL Journal, 2012]

Example: Dependence on ν

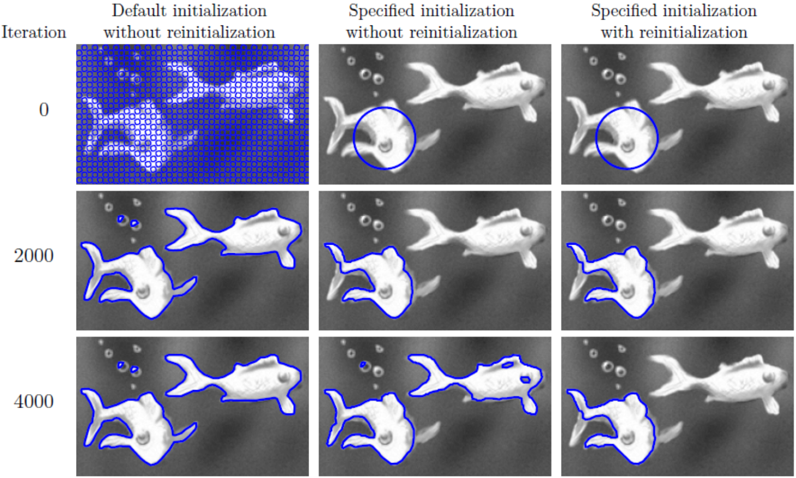


$$\tau = 0.5, \lambda_1 = 1, \lambda_2 = 1,$$
$$\mu = 0.2, \epsilon = 1$$

From [P. Getreuer, Chan–Vese Segmentation, IPOL Journal, 2012]

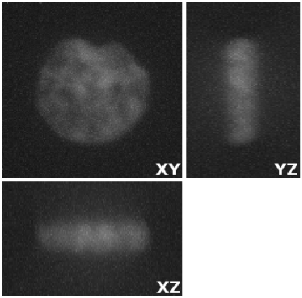
Example: Reinitialization

- Reinitialization is required to avoid separated objects

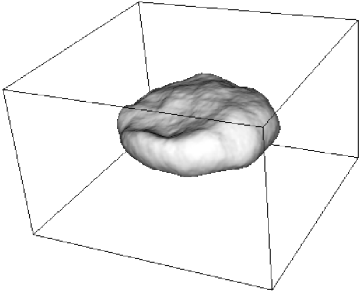


$$\tau = 0.5, \lambda_1 = 1, \lambda_2 = 1,$$
$$\mu = 0.15, \nu = 0, \epsilon = 1$$

3D Example



Input
Size: 160 × 150 × 60



Result

Contents

Summary

- Mumford-Shah functional is a general variational formulation of segmentation
  - It is mathematically very difficult (search for function u and contours $\mathcal{C}$)
  - Non-convex
  - Computationally expensive
- Chan-Vese active contours without edges is a region based segmentation approach
  - Segmentation into 2 components
  - Considers homegenity of regions not only contours
  - It is a special case of Mumfird-Shah functional
  - Fast computation based on level sets

Contents

References

- Getreuer, Pascal. "Chan-vese segmentation."Image Processing On Line 2 (2012): 214-224.
- Chan, Tony F., and Luminita A. Vese. "Active contours without edges."IEEE Transactions on image processing 10.2 (2001): 266-277.