Level Set Methods: Numerical Schemes

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Level Set Methods Recap

Level Set Methods: Numerical Schemes
- Motion in the External Velocity Field
- Mean Curvature Motion
- Motion in the Normal Direction
- All Types of Motion Together

Geodesic Active Contours
- Evolution Equation
- Solution

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Active Contour Model

- The deformable contour (snake) is a mapping:

$$\mathcal{C}(s) : [0,1] \to \mathbb{R}^2, \quad s \mapsto \mathcal{C}(s) = (x(s), y(s))^T.$$

- We define energy functional of the contour as

$$E_{snake}(\mathcal{C}) = \int_0^1 E_{int}(\mathcal{C}(s)) + E_{ext}(\mathcal{C}(s))ds,$$

where $E_{int}(\mathcal{C}(s))$ is internal energy defined as

$$E_{int}(\mathcal{C}(s)) = \alpha(s)|\mathcal{C}'(s)|^2 + \beta(s)|\mathcal{C}''(s)|^2$$

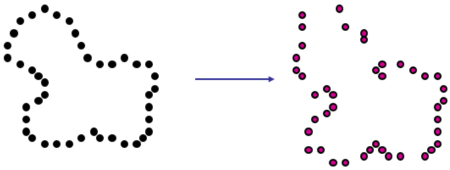
and $E_{ext}(\mathcal{C}(s))$ is external energy defined as

$$E_{ext}(\mathcal{C}(s)) = P(\mathcal{C}(s)),$$

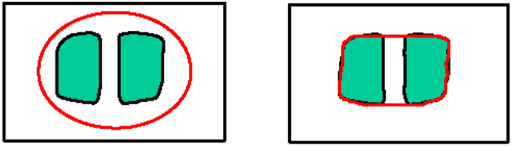
where P is the potential associated to the image.

Problems with Parametric Curves

- Reparametrization needed (hard with surfaces in 3D)



- Cannot handle topological changes



- Hard to extend to 3D

$$\gamma \frac{\partial \mathbf{v}}{\partial t} - \frac{\partial}{\partial s}\left(w_{10}\frac{\partial \mathbf{v}}{\partial s}\right) - \frac{\partial}{\partial r}\left(w_{01}\frac{\partial \mathbf{v}}{\partial r}\right) + 2\frac{\partial^2}{\partial s \partial r}\left(w_{11}\frac{\partial^2 \mathbf{v}}{\partial s \partial r}\right)$$
$$+\frac{\partial^2}{\partial s^2}\left(w_{20}\frac{\partial^2 \mathbf{v}}{\partial s^2}\right) + \frac{\partial^2}{\partial r^2}\left(w_{02}\frac{\partial^2 \mathbf{v}}{\partial r^2}\right) + \nabla P(\mathbf{v}(s,r)) = 0,$$

Level Sets Idea

- Curve $\mathcal{C}$ is represented implicitly as a zero level set of a higher-dimensional function $u : \mathbb{R}^2 \to \mathbb{R}$.

  $$\mathcal{C} = \{(x, y) : u(x, y) = 0\}$$

  In level set formulation, the curve evolution according to

  $$\frac{\partial \mathcal{C}}{\partial t} = \beta \boldsymbol{n}$$

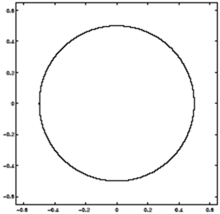
  leads to the evolution of embedding function u according to

  $$\frac{\partial u}{\partial t} + \beta |\nabla u| = 0$$

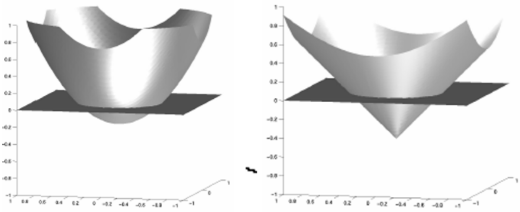
  where $\boldsymbol{n}$ is curve normal and β is the evolution speed (scalar).

Level Sets Idea

- Curve



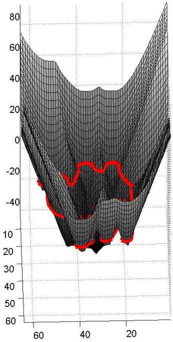
- Different embedding functions $u(x, y)$



Signed Distance Function (SDF): Example

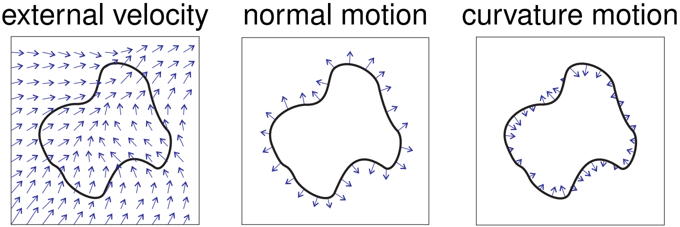
- For $C = \partial\Omega$, SDF d defined by:
  $$d(\mathbf{x}) = \begin{cases} -\min_{\mathbf{y}\in C} |\mathbf{x}-\mathbf{y}| & \text{if } \mathbf{x} \in \Omega^- \\ +\min_{\mathbf{y}\in C} |\mathbf{x}-\mathbf{y}| & \text{if } \mathbf{x} \in \Omega^+ \cup C \end{cases}$$
- Normal $n = \nabla d$, $|\nabla d| = 1$
- Curvature $\kappa = \nabla \cdot \nabla d = \nabla^2 d = \Delta d$

digital shape



SDF $d(\mathbf{x})$

Three types of motion



Level Set Evolution Equations

- Contour motion is driven by PDE

$$u_t + \beta |\nabla u| = 0$$

  where β is the rate of evolution in the normal direction to the contour.
- There are three basic types of motion:
  - motion in the external velocity field $V(x, y, t)$

    $$\beta = V(x, y, t) \cdot n$$

  - motion in the normal direction (balloon force, dilation).

    $$\beta = a$$

  - motion involving mean curvature (internal force)

    $$\beta = -\epsilon\kappa$$

- Different motions require different numerical schemes.

Level Set Evolution Equations: Normal Motion

- Normal motion:

$$u_t(t, \mathbf{x}) + a(\mathbf{x})|\nabla u(t, \mathbf{x})| = 0$$

$$u(0, \mathbf{x}) = u_0(\mathbf{x})$$

- If $a(\mathbf{x}) > 0$ we can transform to boundary value problem: Crossing times $T(\mathbf{x})$ of zero level set in all points by solving the Eikonal equation

$$a(\mathbf{x})|\nabla T(\mathbf{x})| = 1\,,$$
$$T(C_0) = 0, \text{ where } C_0 \text{ is initial contour}$$

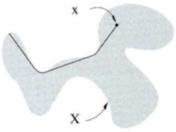
- Efficient numerical algorithm — Fast Marching Method ($\mathcal{O}(n \log n)$)

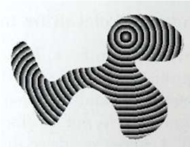
Normal Motion: Choice of $a(\mathbf{x})$

- Euclidean Distance (or SDF): $a(\mathbf{x}) = 1$



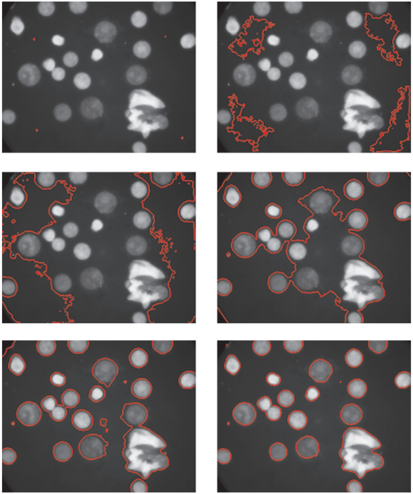
- Geodesic distance: $a(\mathbf{x}) = 1$ inside a mask and $a(\mathbf{x}) \to 0$ outside the mask





Normal Motion: Choice of $a(\mathbf{x})$

- Segmentation: $a(\mathbf{x}) = g(|\nabla f(\mathbf{x})|)$, where g is a decreasing function, e.g. $g = 1/(1 + \lambda |\nabla G_\sigma * f(\mathbf{x})|)$



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$$\beta = -\epsilon\kappa$$

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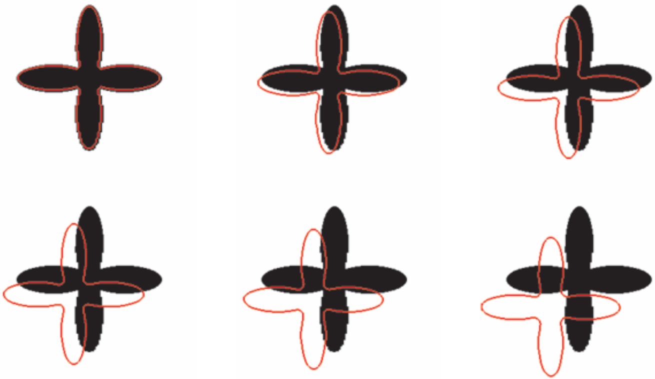
Motion in the external velocity field



Motion in the external velocity field. The velocity vector is equal to $(-1, -1)$ for every grid point in this example.

Motion in the External Velocity Field

Suppose an (external) velocity field $V(x, y) = (v(x, y), w(x, y))$. Speed in normal direction is given by $\beta = V \cdot n$. Therefore, we have

$$u_t + (V \cdot n)|\nabla u| = u_t + V \cdot \frac{\nabla u}{|\nabla u|}|\nabla u| = 0$$

Then the evolution of interface embedded in implicit function u

is described by convection (or advection) equation:

$$u_t + V \cdot \nabla u = 0$$

or we can write

$$u_t + vu_x + wu_y = 0$$

Numerical Solution

$$u_t + vu_x + wu_y = 0$$

- Time discretization – forward Euler

  $u_t = \frac{u_i^{n+1} - u_i^n}{\tau}$

- Spatial discretization – upwind differencing dimension-by-dimension (shown for x axis)

  if $v > 0$ then $u_x = \frac{u_i^n - u_{i-1}^n}{h}$ notation D^{-x}

  if $v < 0$ then $u_x = \frac{u_{i+1}^n - u_i^n}{h}$ notation D^{+x}

  if $v = 0$ then 0

Numerical Solution

$$u_t + vu_x + wu_y = 0$$

has iterative numerical scheme

$$u_{ij}^{n+1} = u_{ij}^n - \tau \left(\begin{array}{l} \min(v_{ij}, 0) D_{ij}^{+x} + \max(v_{ij}, 0) D_{ij}^{-x} + \\ \min(w_{ij}, 0) D_{ij}^{+y} + \max(w_{ij}, 0) D_{ij}^{-y} \end{array} \right),$$

where

$$D_{ij}^{+x} = \frac{u_{i+1,j}^n - u_{ij}^n}{h}, \quad D_{ij}^{-x} = \frac{u_{ij}^n - u_{i-1,j}^n}{h}$$

$$D_{ij}^{+y} = \frac{u_{i,j+1}^n - u_{ij}^n}{h}, \quad D_{ij}^{-y} = \frac{u_{ij}^n - u_{i,j-1}^n}{h}$$

Higher order schemes exists.

Stability condition: $h/\tau > \max\{v, w\}$ for all grid points.

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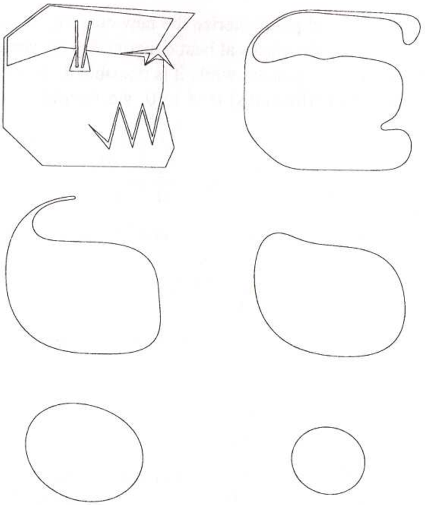
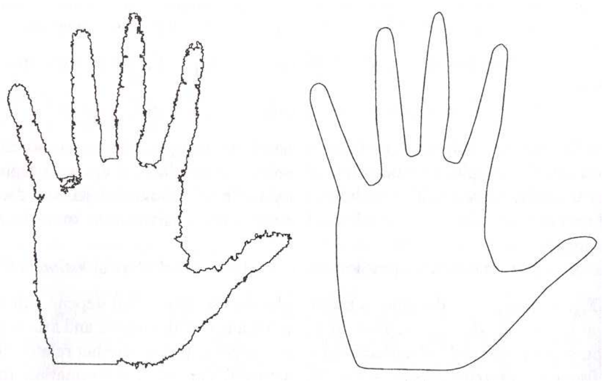
Curvature Driven Motion, Example



Shape Smoothing



Mean Curvature Motion

Mean curvature motion ($\epsilon = 1.0$)

Curve Length and Shortening Flow

- Length of parametric curve can be computed as

$$L(\mathcal{C}) = \int_0^1 |\mathcal{C}'(q)| dq = \int_{\mathcal{C}} ds$$

  where $|\mathcal{C}'(q)|$ is parametrization speed and ds is arc-length.
- It can be proved that the equation

$$\frac{\partial \mathcal{C}}{\partial t} = \kappa \mathbf{n},$$

  where κ is curvature and $\mathbf{n}$ is curve normal, gives the fastest way to reduce L, i.e., moves the curve in the direction of the gradient of the functional L.
- This equation is known as the Euclidean curve shortening flow.

Mean Curvature Motion

Mean curvature motion is given by the speed $\beta = -\epsilon\kappa$, where $\epsilon > 0$ is constant and κ is the curvature.

Then the evolution equation is:

$$u_t = \epsilon\kappa|\nabla u|$$

Numerical scheme:

- Time discretization – forward Euler (first order)

  $u_t = \frac{u_i^{n+1} - u_i^n}{\tau}$

- Spatial discretization – central differencing (second order) both for the gradient ∇u and the curvature $\kappa = \nabla \mathbf{n} = (u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx})/|\nabla u|^3$

  $u_x = \frac{u_{i+1}^n - u_{i-1}^n}{2h}$ notation D^{0x}

Numerical Solution

Equation

$$u_t = \epsilon\kappa|\nabla u|$$

has iterative numerical scheme

$$u_{ij}^{n+1} = u_{ij}^{n} + \tau(\epsilon K_{ij}^{n}\sqrt{(D_{ij}^{0x})^2 + (D_{ij}^{0y})^2}),$$

where K_{ij}^{n} is central difference approximation to κ in point (x_i, y_j) and time $n\tau$.

Stability condition: $4\tau\epsilon < h^2$. Therefore, τ needs to be $\mathcal{O}(h^2)$, i.e. it is smaller (one order of magnitude) than in the previous case.

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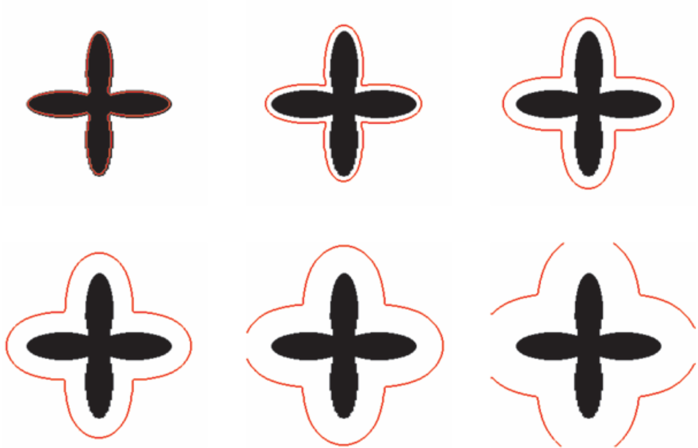
Motion in the Normal Direction



Motion in the normal direction. $a = 1.0$.

Motion in the Normal Direction

Internally generated velocity field $\beta = a$
Then the evolution of u is described by the equation:

$$u_t + a|\nabla u| = 0$$

which leads to the numerical scheme

$$u_{ij}^{n+1} = u_{ij}^{n} - \tau[\max(a,0)\nabla^+ + \min(a,0)\nabla^-],$$

where

$$\nabla^+ = \left[\begin{array}{l} \min(D_{ij}^{+x},0)^2 + \max(D_{ij}^{-x},0)^2 + \\ \min(D_{ij}^{+y},0)^2 + \max(D_{ij}^{-y},0)^2 \end{array} \right]^{1/2}$$

$$\nabla^- = \left[\begin{array}{l} \max(D_{ij}^{+x},0)^2 + \min(D_{ij}^{-x},0)^2 + \\ \max(D_{ij}^{+y},0)^2 + \min(D_{ij}^{-y},0)^2 \end{array} \right]^{1/2}.$$

Stability condition: $\tau a < h$

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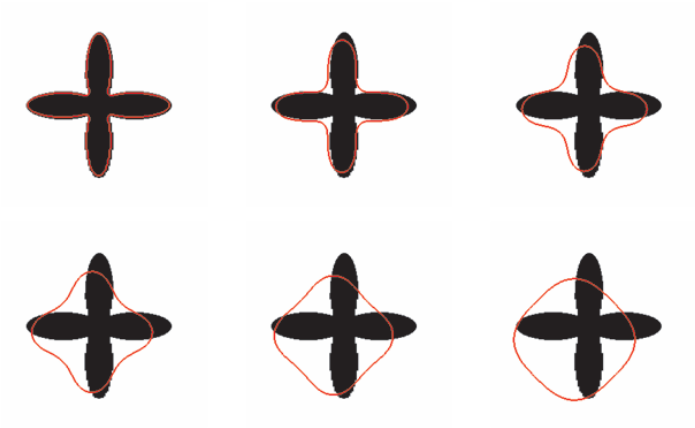
All Types of Motion Together



All three types of motion together. All vectors in V are equal to $(-1, -1)$, $\epsilon = 0.25$ and $a = 1.0$.

All Types of Motion Together

The general equation

$$u_t + V \cdot \nabla u + a|\nabla u| = \epsilon\kappa|\nabla u|$$

is discretized with the numerical scheme

$$u_{ij}^{n+1} = u_{ij}^{n} - \tau \left[\begin{array}{c} [\max(a,0)\nabla^{+} + \min(a,0)\nabla^{-}] + \\ \\ + \left[\begin{array}{c} \max(w_{ij}^{n},0)D_{ij}^{-x} + \min(w_{ij}^{n},0)D_{ij}^{+x} \\ + \max(v_{ij}^{n},0)D_{ij}^{-y} + \min(v_{ij}^{n},0)D_{ij}^{+y} \end{array} \right] - \\ \\ -[\epsilon K_{ij}^{n}\sqrt{(D_{ij}^{0x})^2 + (D_{ij}^{0y})^2}] \end{array} \right],$$

Note: It's a combination of previous formulas.

Stability condition: The most restrictive (curvature) term forces $\tau = \mathcal{O}(h^2)$

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Active Contours: Snakes

- Active contour model (snakes) can be written as:

$$E_{AC}(\mathcal{C}) = \int_0^1 \left(\alpha|\mathcal{C}'(q)|^2 + \beta|\mathcal{C}''(q)|^2\right) dq + \int_0^1 P(\mathcal{C}(q))dq,$$

  where P is the potential field.
- Often $P(\mathcal{C}(q)) = g(|\nabla G_\sigma * I(\mathcal{C}(q))|)$ where $g : [0, \infty) \to \mathbb{R}^+$ is a strictly decreasing function such that $g(r) \to 0$ as $r \to \infty$.
- Problem: (internal) energy depends on curve parametrization, i.e. it changes if we substitute $q = \phi(r)$, $\phi : [c, d] \to [0, 1]$, $\phi' > 0$
- This is an undesirable property, since parametrizations are not related to the geometry of the curve (or object boundary), but only to the velocity they are traveled.

Geodesic Active Contours

- Caselles et.al. 1995, Kichenassamy et al. 1995
- Curve with minimal geodesic length is searched.

$$E_{GAC}(\mathcal{C}) = \int_0^{L(\mathcal{C})} g(|\nabla G_\sigma * I(\mathcal{C}(s))|)ds =$$
$$= \int_0^1 g(|\nabla G_\sigma * I(\mathcal{C}(q))|)|\mathcal{C}'(q)|dq$$

where $g : [0, \infty) \to \mathbb{R}^+$ is a strictly decreasing function such that $g(r) \to 0$ as $r \to \infty$, e.g.,

$$g(|\nabla G_\sigma * I(x, y)|) = \frac{1}{1 + |\nabla G_\sigma * I(x, y)|}$$

- The curve is attracted by image edges, where the weight $g(|\nabla G_\sigma * I(C(q))|)$ is small.
- Energy functional is not convex and therefore there are several local minima.

Geodesic Active Contours Advantages

- Do not depend on parametrization of curve $\mathcal{C}$
- Allow topology changes
- Curve shortening flow is embeded
- Fast computation with level sets
- Straightforward extension to 3D

Geodesic Active Contours, Evolution Equation

- Evolution equation (i.e. Euler-Lagrange equation with $\frac{\partial \mathcal{C}}{\partial t}$ on the left side) for geodesic active contours is

$$\frac{\partial \mathcal{C}}{\partial t} = (g\kappa - (\nabla g \cdot \mathbf{n}))\,\mathbf{n}$$

  where **n** is curve normal (vector) and κ is curvature (scalar) [see E. Sakhaee, "A Tutorial on Active Contours", 2014]
- This equation can be rewritten in level-set framework.
- The curve is embeded in SDF u with evolution speed $\nu = g\kappa - \nabla g \cdot n$.
  The evolution equation becomes

$$\frac{\partial u}{\partial t} = g\kappa|\nabla u| - \nabla g \cdot \nabla u$$

Geodesic Active Contours, Evolution Equation

- Curve with minimal geodesic length is searched.

$$E_{GAC}(\mathcal{C}) = \int_0^1 g(|\nabla G_\sigma * I(\mathcal{C}(q))|)|\mathcal{C}'(q)|dq$$

$$g(|\nabla G_\sigma * I(x,y)|) = \frac{1}{1 + |\nabla G_\sigma * I(x,y)|}$$

- The evolution equation becomes

$$\frac{\partial u}{\partial t} = g\kappa|\nabla u| - \nabla g \cdot \nabla u$$

- Illustration

I $\quad$ g $\quad$ $-\nabla g$

Geodesic Active Contours, Evolution Equation

- The evolution equation

$$\frac{\partial u}{\partial t} = g\kappa|\nabla u| - \nabla g \cdot \nabla u$$

  contains curvature motion and velocity field motion.
- It is solved using level set methods
- We can add normal direction motion (baloon force)

$$\frac{\partial u}{\partial t} = (c + \kappa)g|\nabla u| - \nabla g \cdot \nabla u$$

Geodesic Active Contours, Evolution Equation

- The most general form

$$\frac{\partial u}{\partial t} = (c + \epsilon\kappa)g|\nabla u| + \beta(\nabla P \cdot \nabla u)$$

where $P = |\nabla G_\sigma * I|$, c is a constant and

$$g(|\nabla G_\sigma * I(x, y)|) = \frac{1}{1 + |\nabla G_\sigma * I(x, y)|^p}$$

- Parameters:
ϵ, β, c, σ and p

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Geodesic Active Contours, Iterative Scheme

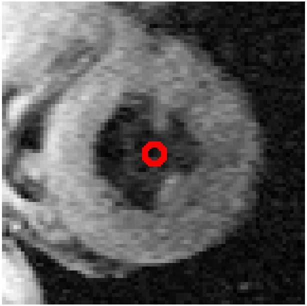
- The equation

$$u_t = (c + \epsilon\kappa) g |\nabla u| + \beta \nabla P \cdot \nabla u$$

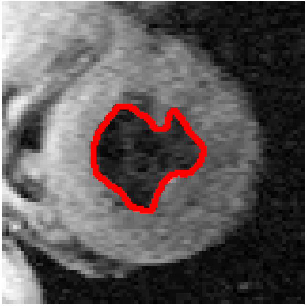
  consists of three types of motion we have discussed before:
  - normal direction motion with speed $cg(|\nabla G_\sigma * I(x, y)|)$
  - curvature motion multilied by a factor $\epsilon g(|\nabla G_\sigma * I(x, y)|)$
  - external velocity field motion given by $\beta \nabla P$.
- Therefore, we have

$$u_{ij}^{k+1} = u_{ij}^{k} + \tau \cdot [\text{Normal}(cg) + \text{Curvature}(\epsilon g) + \text{Velocity}(\beta \nabla P)].$$

Geodesic Active Contours: Example



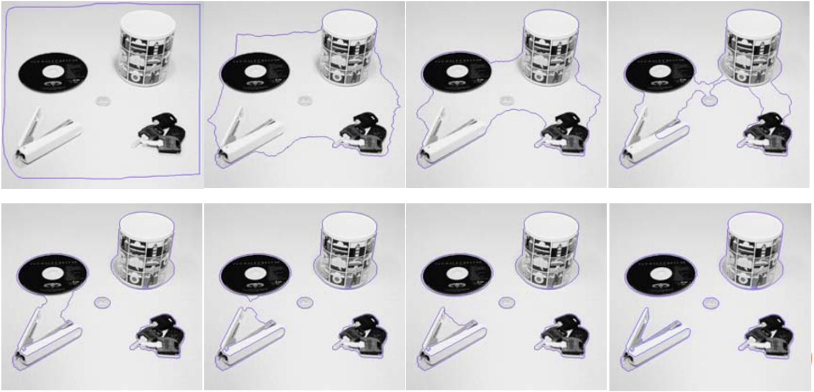
Initial contour



Result

$\tau = 0.25, c = 1, \epsilon = 0.5,$
$\beta = 0.5, p = 2, \sigma = 2.0$

Geodesic Active Contours: Example



Geodesic Active Contours: Example

Geodesic Active Contours: Example

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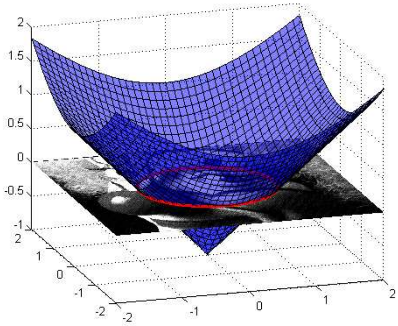
Introduction

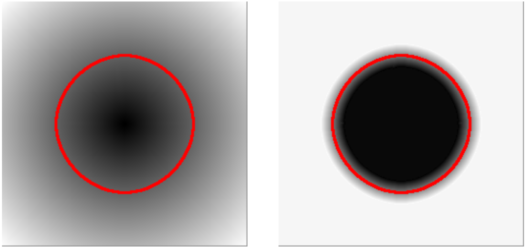
- Contour of interest $\mathcal{C}$ (surface or curve) is the (zero) level set of an implicit function u, i.e. $C = \{x : u(x) = 0\}$
- The evolution is steered by level set methods modifying u.



- We don't need to recompute u in the whole domain to track one contour.

General Procedure

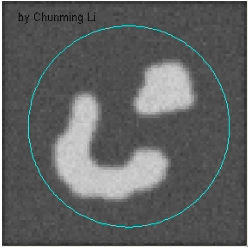
- Embed initial curve into implicit function u as its zero level set. One can compute signed distance function using fast marching algorithm to get u for any contour.
- Solve the evolution equation in a narrow band around the contour of interest only.



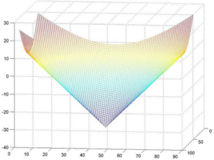
Whole domain   Narrow band

- We have to reinitialize the narrow band regularly.

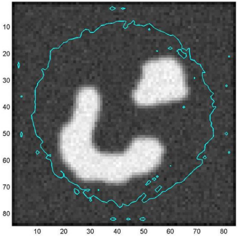
Reinitialization



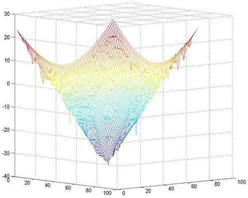
initial image



SDF



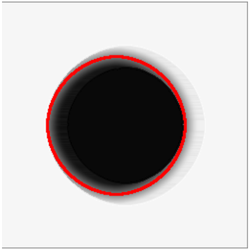
after 50 iterations



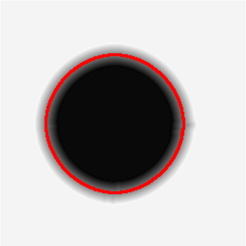
SDF

Narrow Band Reinitialization

- Before the contour leaves the narrow band we must reinitialize it.



Before reinitialization



After reinitialization

- Narrow band reinitialization procedure can be based on the fast marching algorithm.
- Maintaining sign distance function (SDF) around the contour simplifies numerical formulas.

Simplifications of Formulas

- Maintaining sign distance function around the contour simplifies numerical formulas.
- For SDF we have $|\nabla u| = 1$ and therefore the normal is

$$\mathbf{n}(\mathbf{x}) = \frac{\nabla u}{|\nabla u|} = \nabla u$$

and the curvature is

$$\kappa = \nabla \cdot \mathbf{n} = \nabla \cdot \nabla u = \nabla^2 u = \Delta u,$$

i.e. it is simply the Laplacian of u, i.e.
$\kappa(\mathbf{x}) = \Delta u = u_{xx} + u_{yy} + u_{zz}$.

Time complexity

- Narrow band method significantly reduces time complexity of the evolution procedure.
- Instead of $\mathcal{O}(n^{dim})$, where n is the number of grid points in one dimension and dim is the number of dimensions, we have time complexity $\mathcal{O}(kn^{dim-1})$ where k is narrow band width.
- The price is the necessity of regular band reinitialization.

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- All Types of Motion Together

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- Solution

Other Implementation Details

Summary

References

Summary

- We discussed numerical schemes for three different types of motion
- motion in the external velocity field
- mean curvature motion
- motion in the normal direction
- Curvature motion smooths the contour
- Geodesic active contours seeks curve with minimal geodesic length.
- We discussed active contours in the context of level set methods.
- Fast implementations of level set methods handle implicit function around zero level set only (e.g., narrow band method).

Contents

Level Set Methods Recap

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References

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