Convolution

In the spatial domain

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \tilde{f}(x)$$

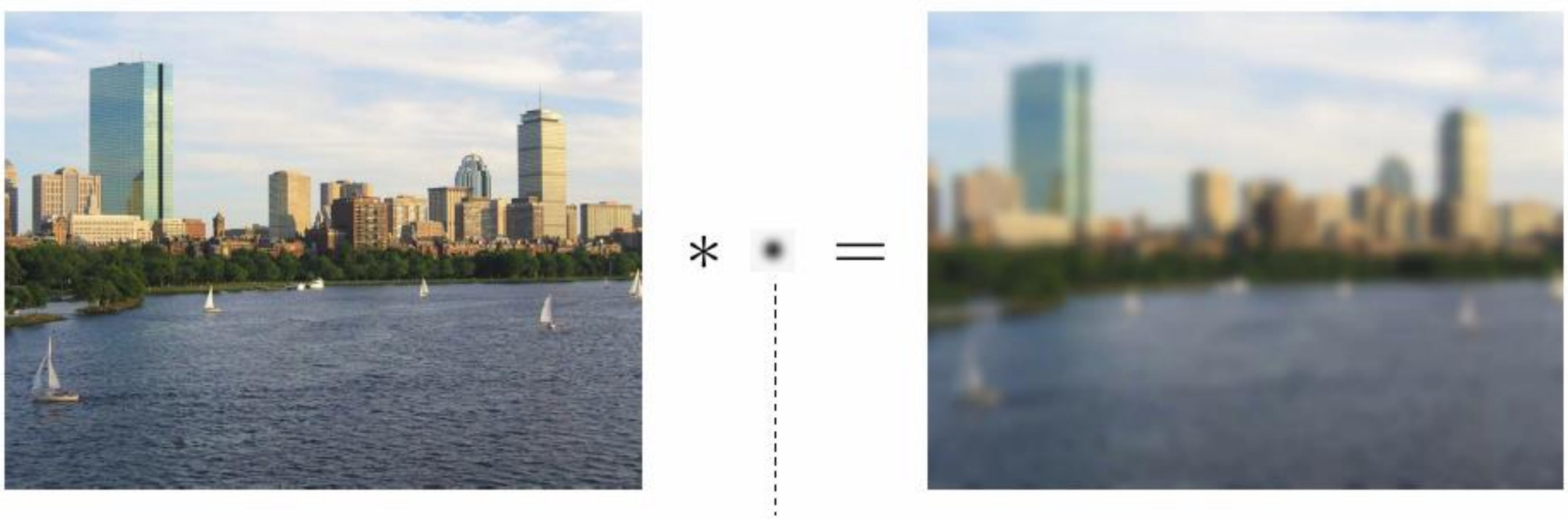
Convolution kernel, filter $g(x)$

Filtered signal $\tilde{f}(x)$

Convolution

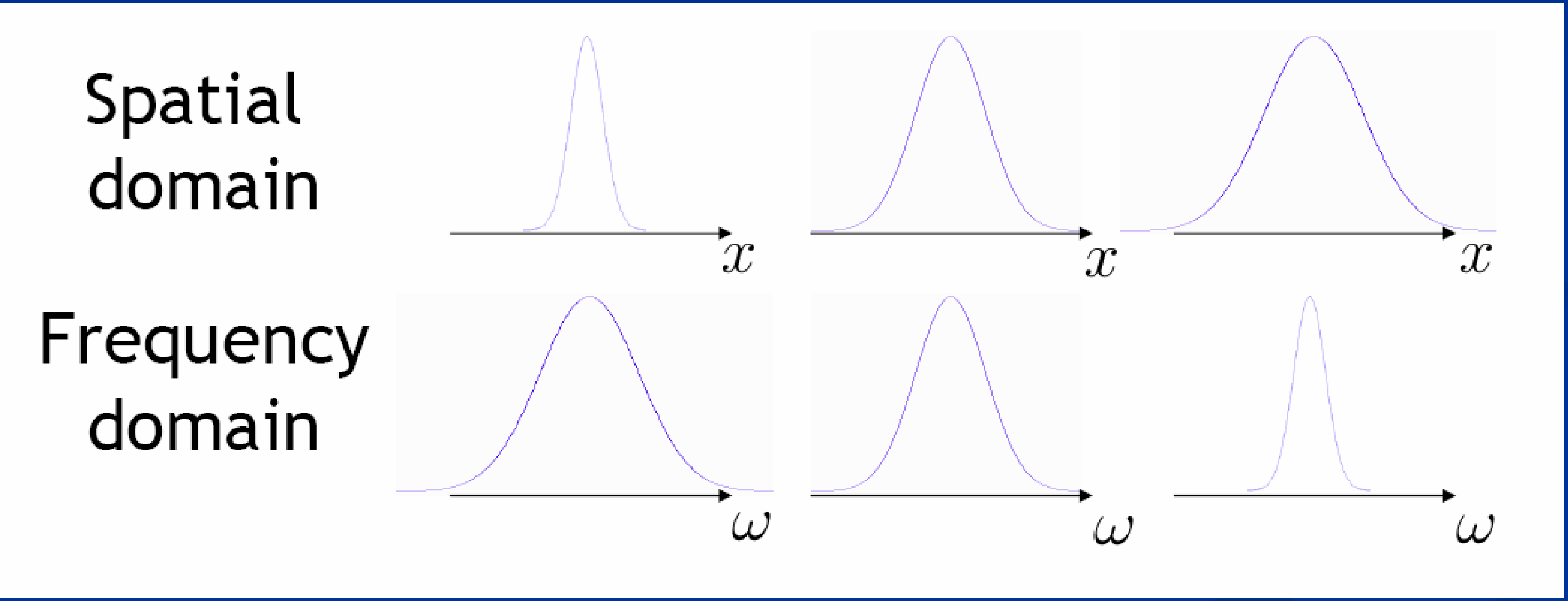
- In the spatial domain

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \tilde{f}(x)$$



$f(x)$ $g(x)$ $f(x) * g(x)$

Convolution



Теорема А.2 (ФУБИНИ). *Если* $\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}|f(x_1,x_2)|dx_1\right)dx_2 < +\infty$, *то*

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_1,x_2)dx_1dx_2 = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(x_1,x_2)dx_1\right)dx_2 = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(x_1,x_2)dx_2\right)dx_1.$$

Теорема о преобразовании Фурье свертки

Теорема. (О свёртке) Пусть $f \in L^1(R)$ и $h \in L^1(R)$. Функция $g = h * f$ принадлежит $L^1(R)$ и

$$\hat{g}(\omega) = \hat{h}(\omega)\hat{f}(\omega) \quad (\hat{g}\text{— преобр. Фурье g})$$

Доказательство:

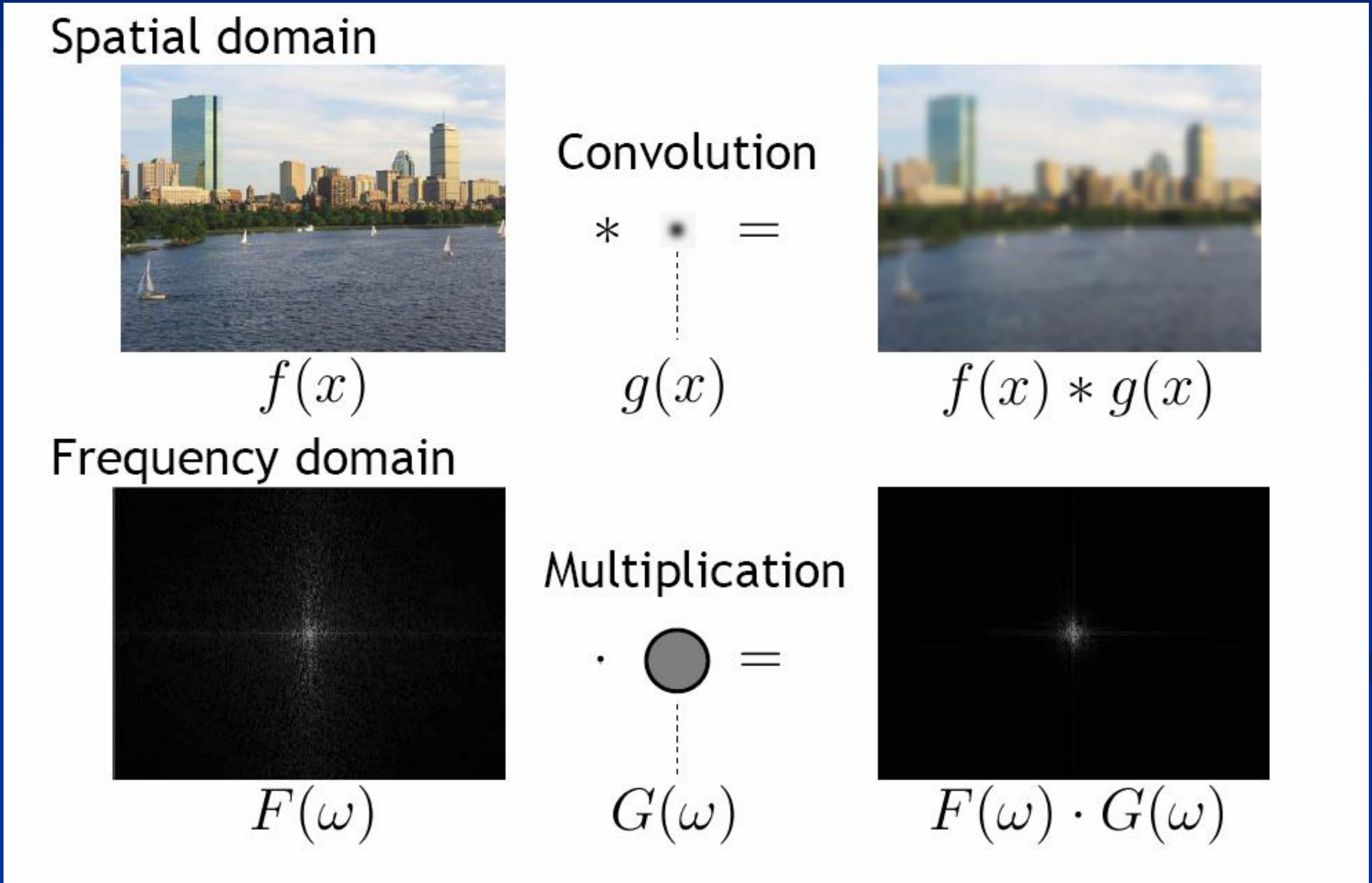
$$\hat{g}(\omega) = \int_{-\infty}^{+\infty} \exp(-it\omega) \left(\int_{-\infty}^{+\infty} f(t-u)h(u)\mathrm{d}u \right) \mathrm{d}t$$

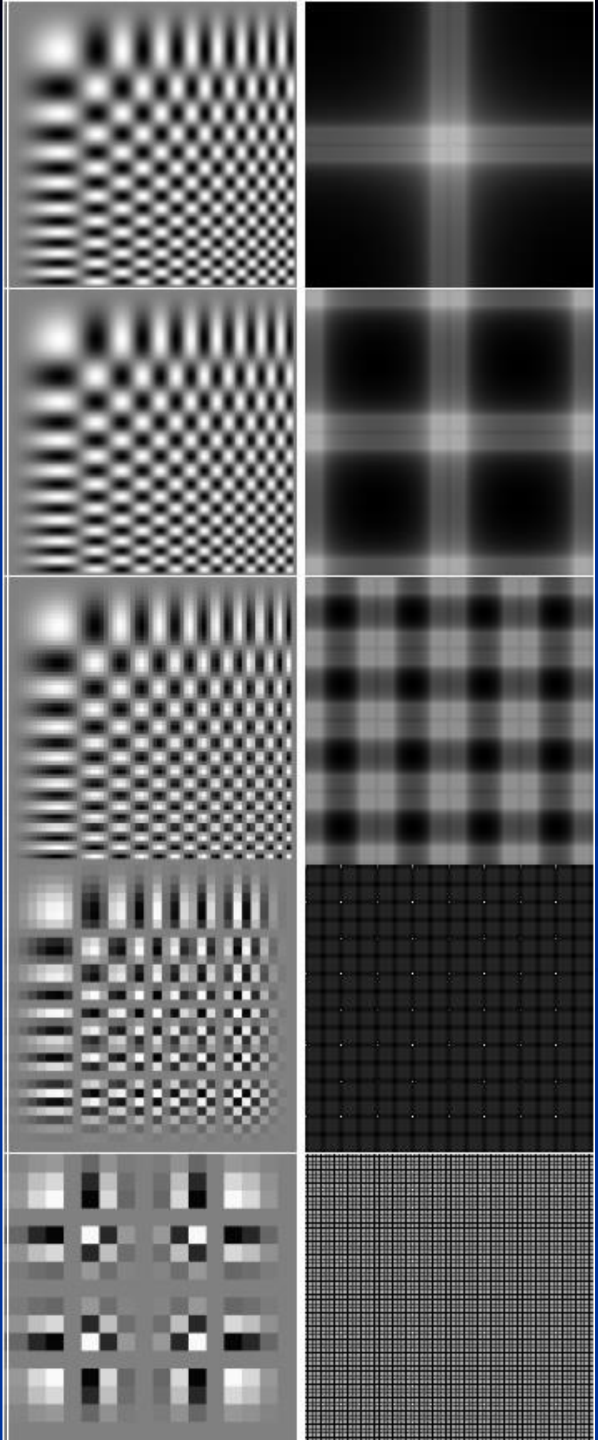
Так как $|f(t-u)||h(u)|$ интегрируема в R^2, мы можем применить теорему Фубини, и замена переменных $(t,u) \to (v = t-u, u)$ даёт

$$\hat{g}(\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp(-i(u+v)\omega) f(v)h(u)\mathrm{d}u\mathrm{d}v = \left(\int_{-\infty}^{+\infty} \exp(-iv\omega) f(v)\mathrm{d}v \right) \left(\int_{-\infty}^{+\infty} \exp(-iu\omega) h(u)\mathrm{d}u \right)$$

теорема доказана.

Convolution





Aliasing →

Dirac delta function

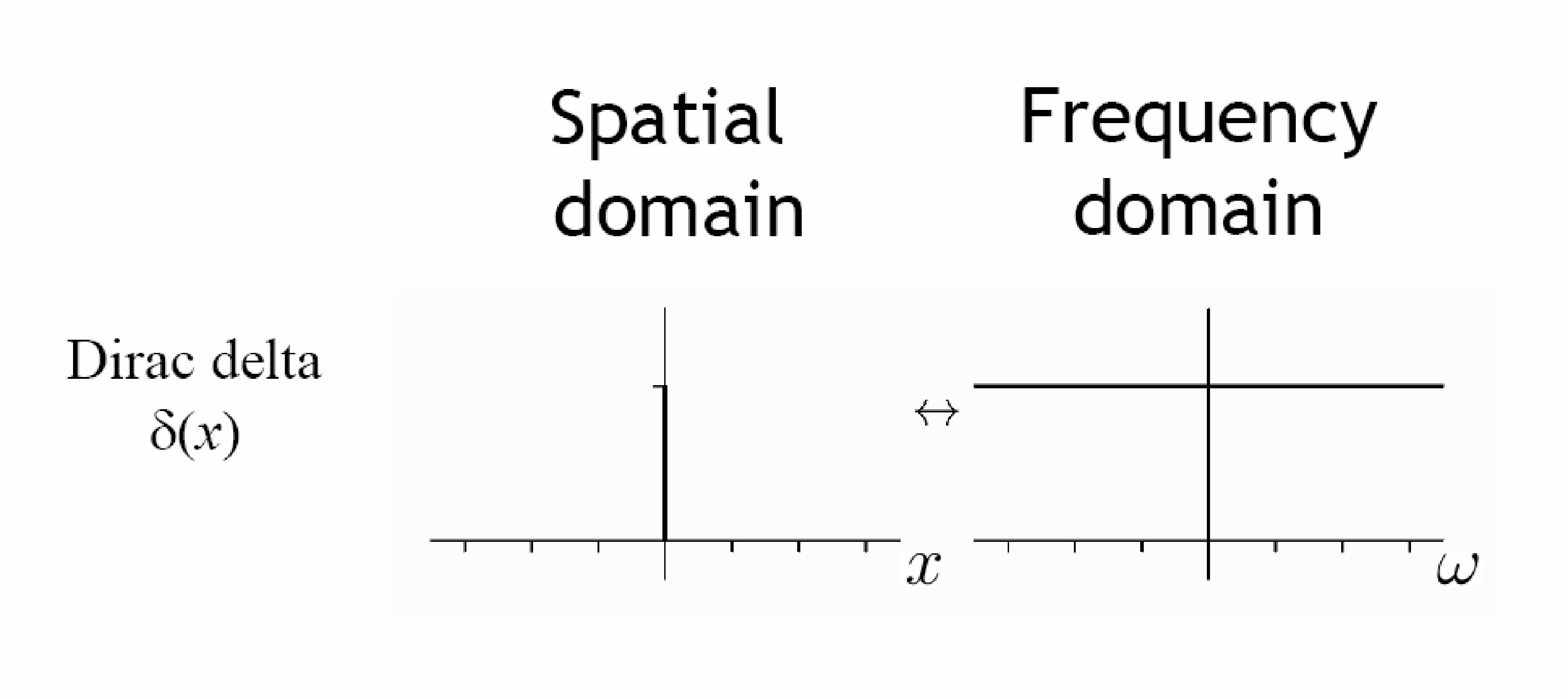
- Definition

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)dx = 1$$

- Sifting property

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\,dx = f(x_0)$$

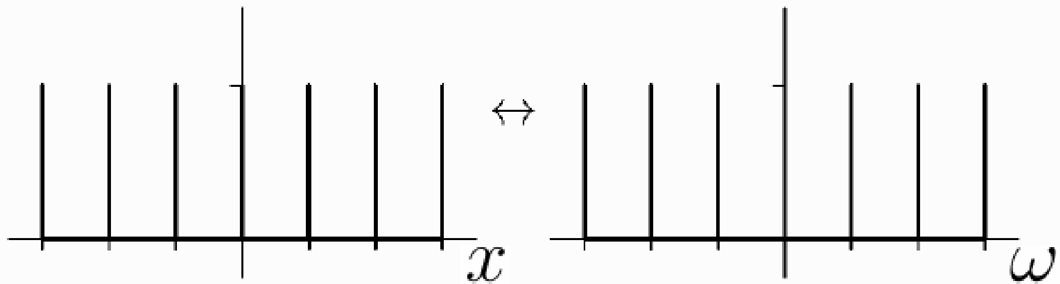
Dirac delta function



Impulse Train

Spatial domain | Frequency domain

Impulse train
(shah, comb function)
$III_T(x)$



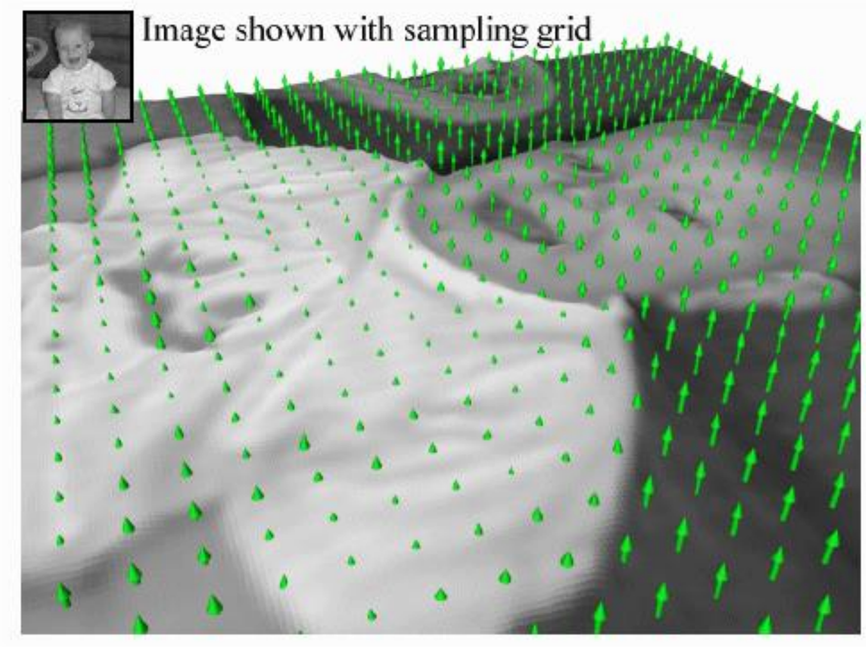
$$III_T(x) = \sum_k \delta(x - kT) \qquad III_{\omega_0}(\omega) = \frac{1}{\sqrt{2\pi\omega_0}} \sum_k \delta(\omega - k\omega_0)$$

Period T | Period $\omega_0 = 2\pi / T$

Sampling

Spatial domain: multiply signal with impulse train

$$f(x) \to f(x) III_T(x)$$



Sampling

Spatial domain: multiply signal with impulse train

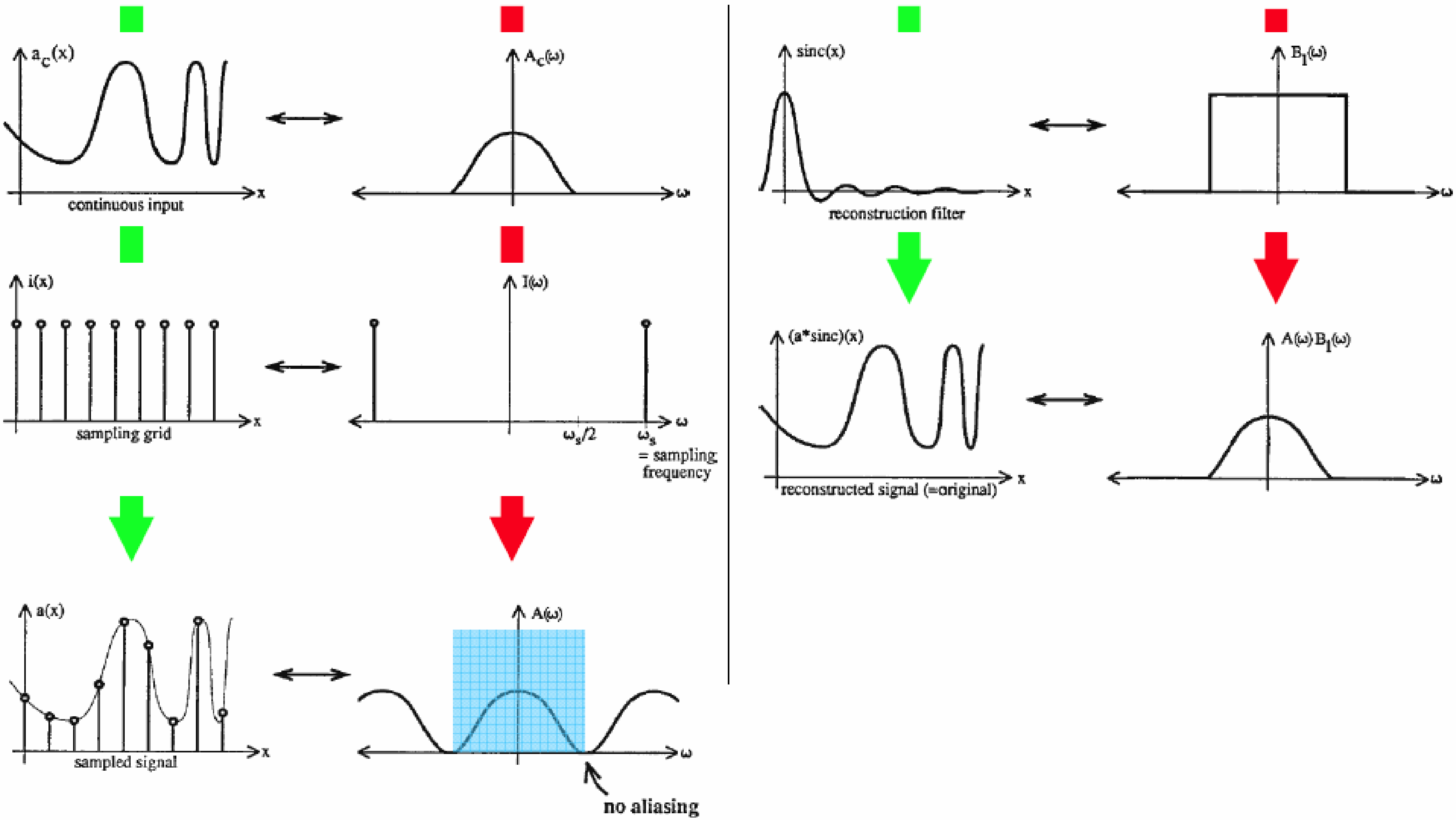
$$f(x) \rightarrow f(x)III_T(x)$$

Frequency domain: convolve signal with Fourier transform of impulse train

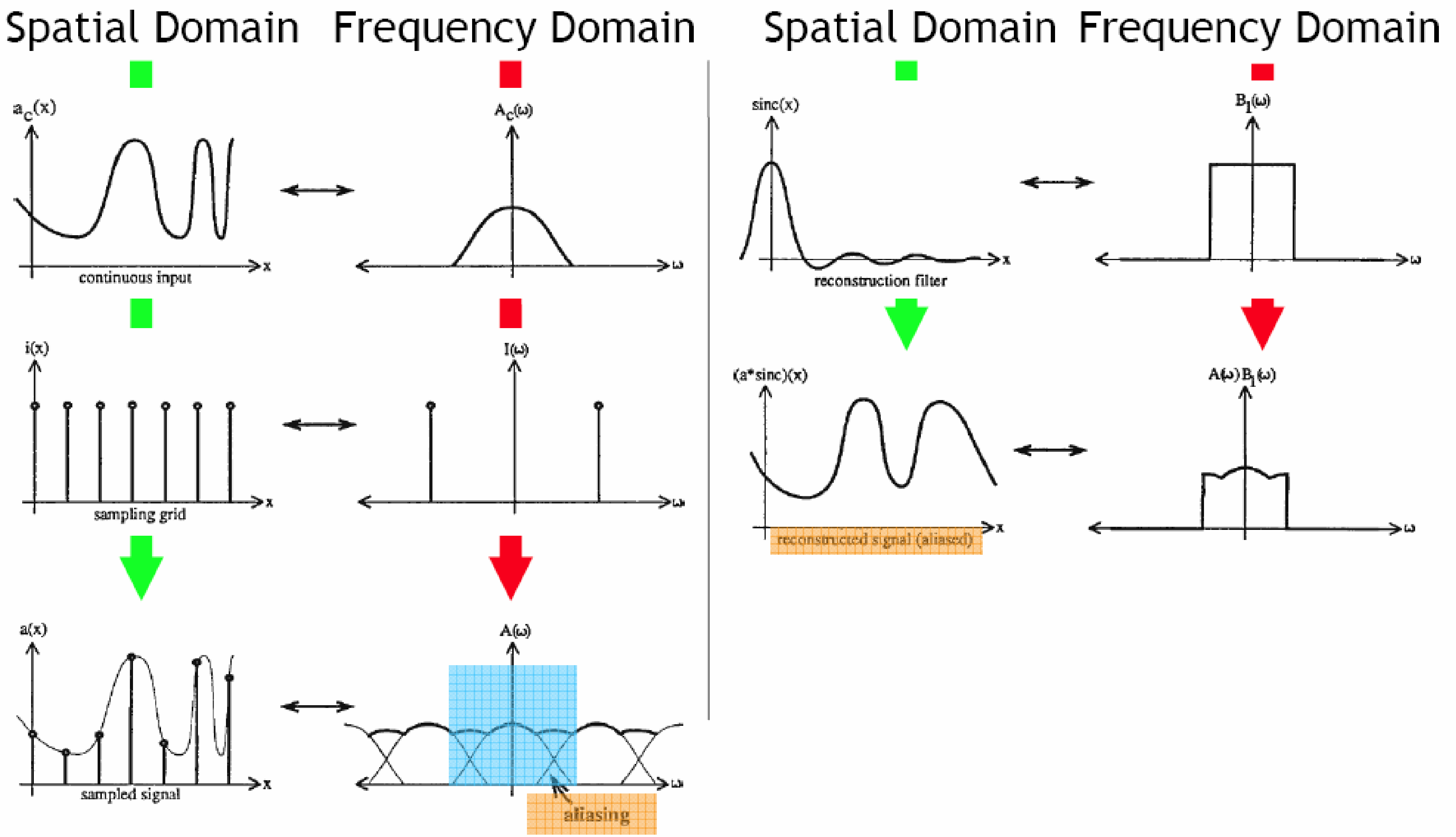
$$F(\omega) \rightarrow F(\omega) * III_{\omega_0}(\omega)$$

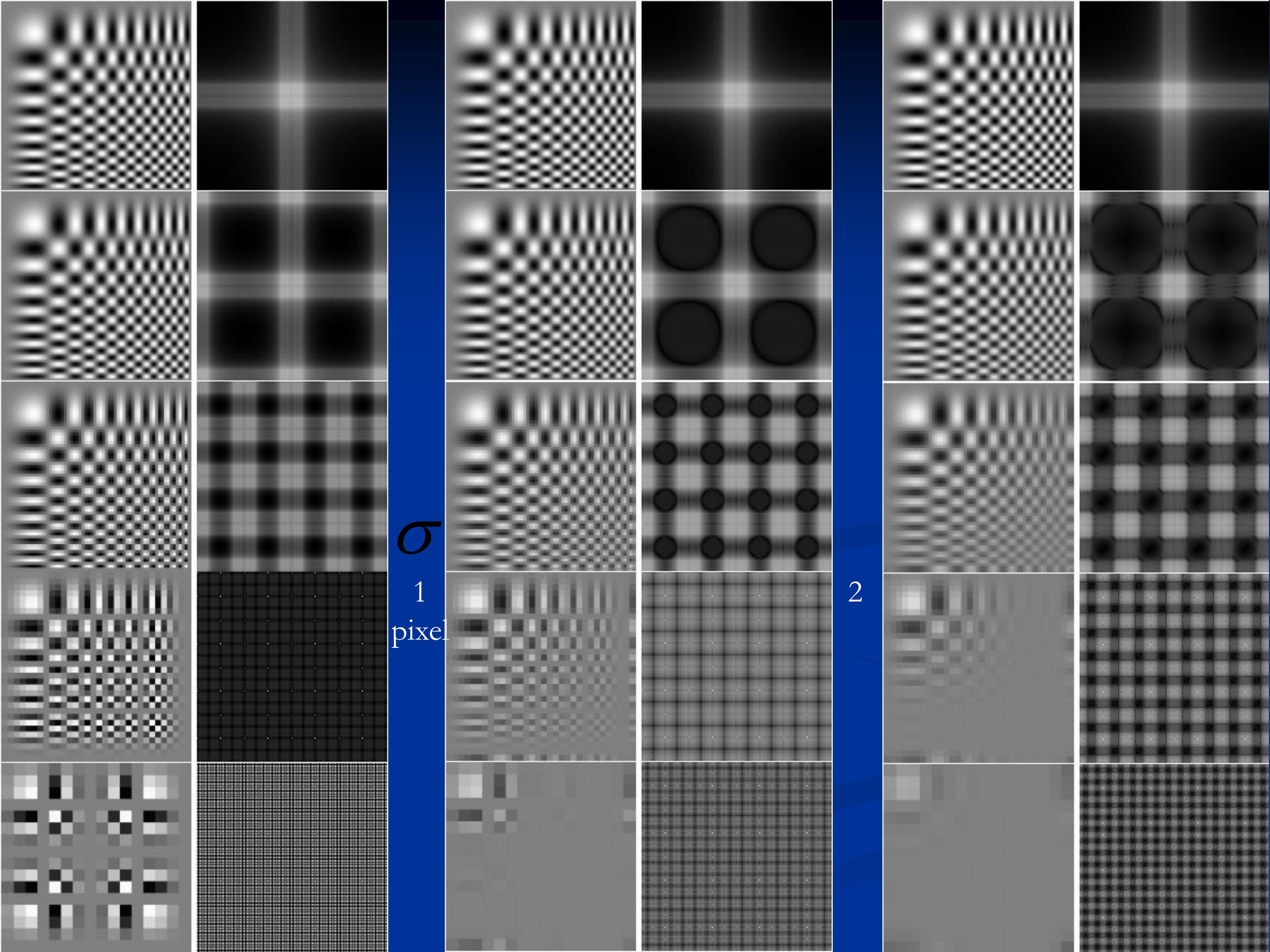
Sampling and reconstruction

Spatial Domain Frequency Domain Spatial Domain Frequency Domain



Sampling and reconstruction

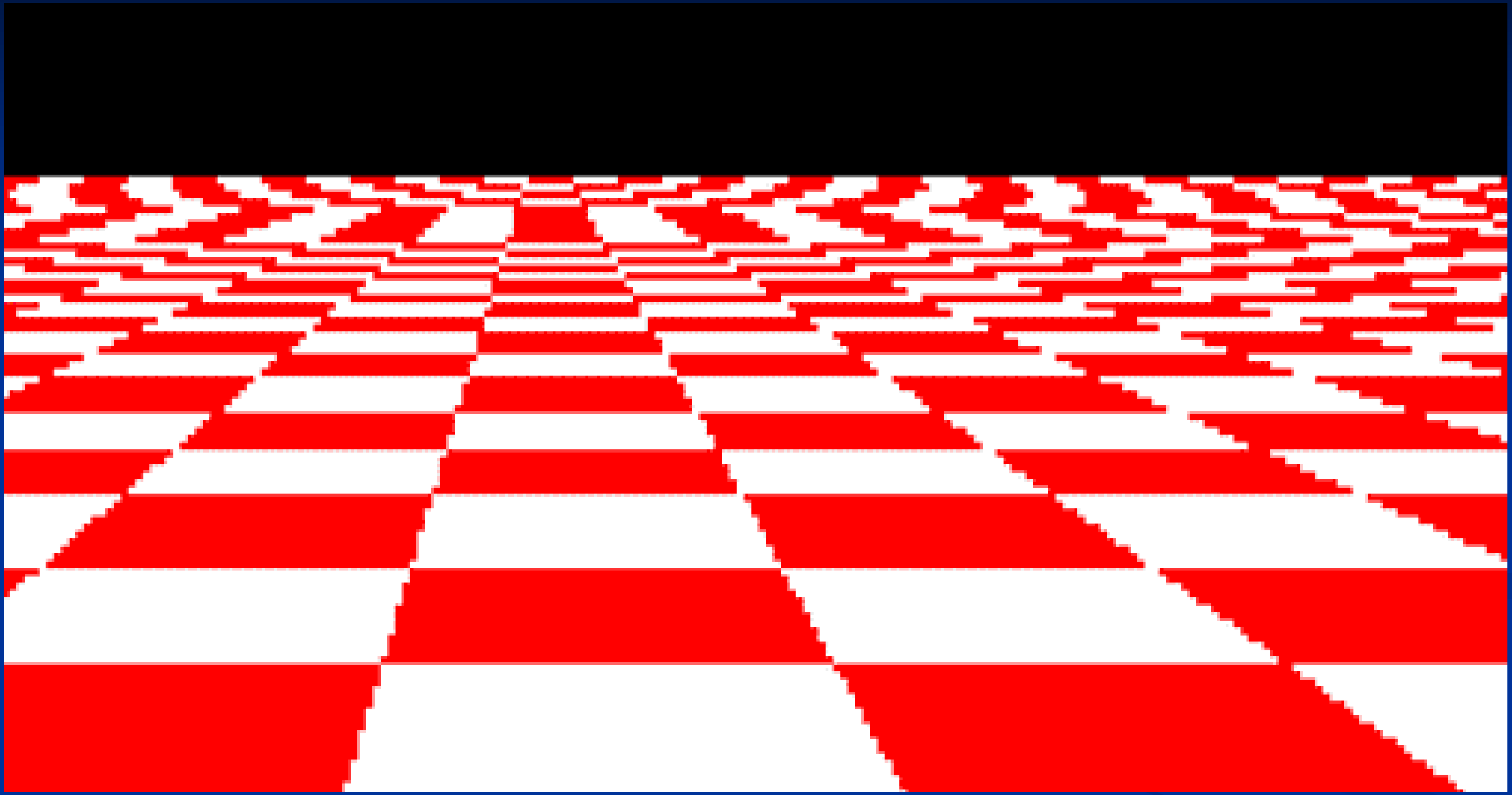



σ
1
pixel
2

Some Aliasing Artifacts

- *Spatial*: Jaggies, Moire
- *Temporal*: Strobe lights, "Wrong" wheel rotations
- *Spatio-Temporal*: Small objects appearing and disappearing

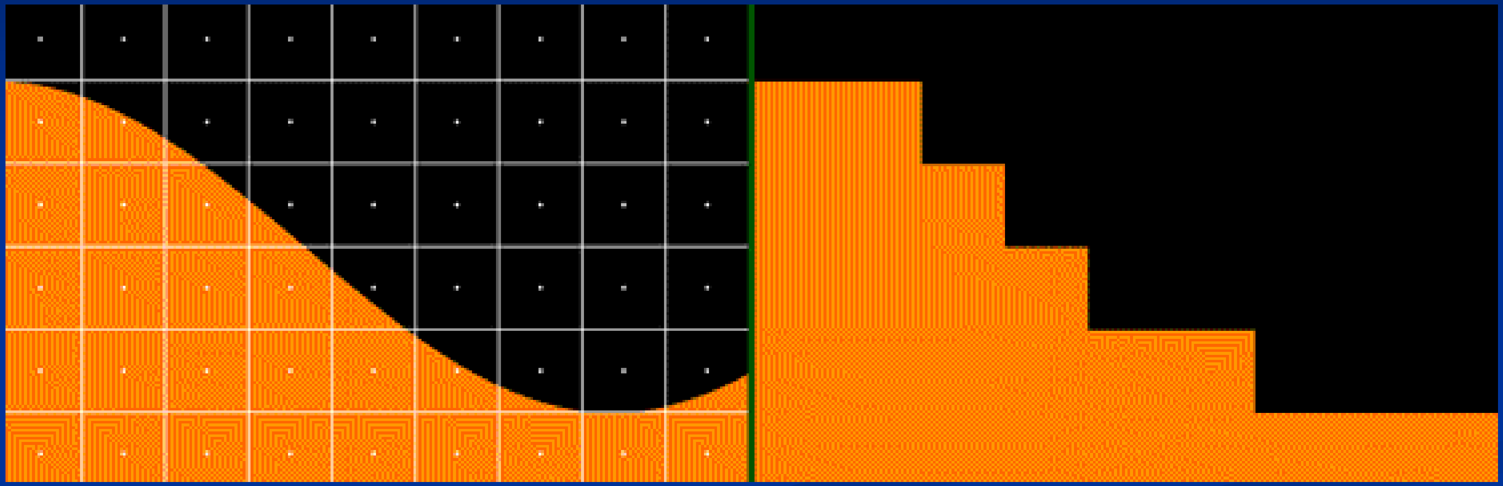
Disintegrating Texture



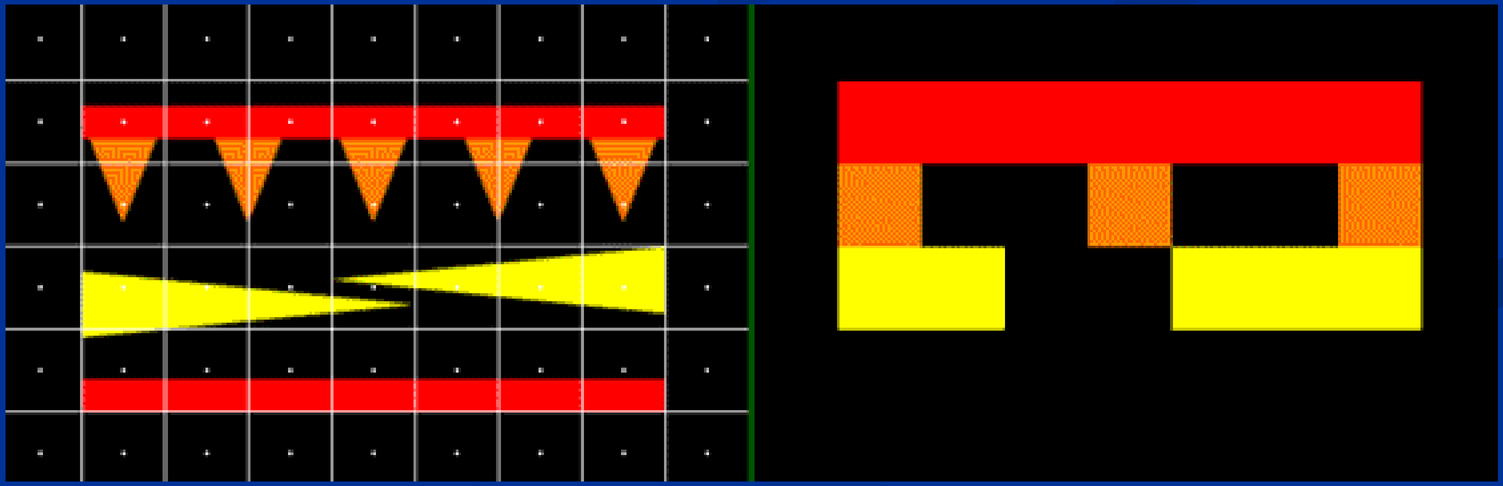
- The checkers on a plane should become smaller with distance.
- But aliasing causes them to become larger and/or irregular.
- Increasing resolution only moves the artefact closer to the horizont.

Spatial Aliasing

Jagged Profiles

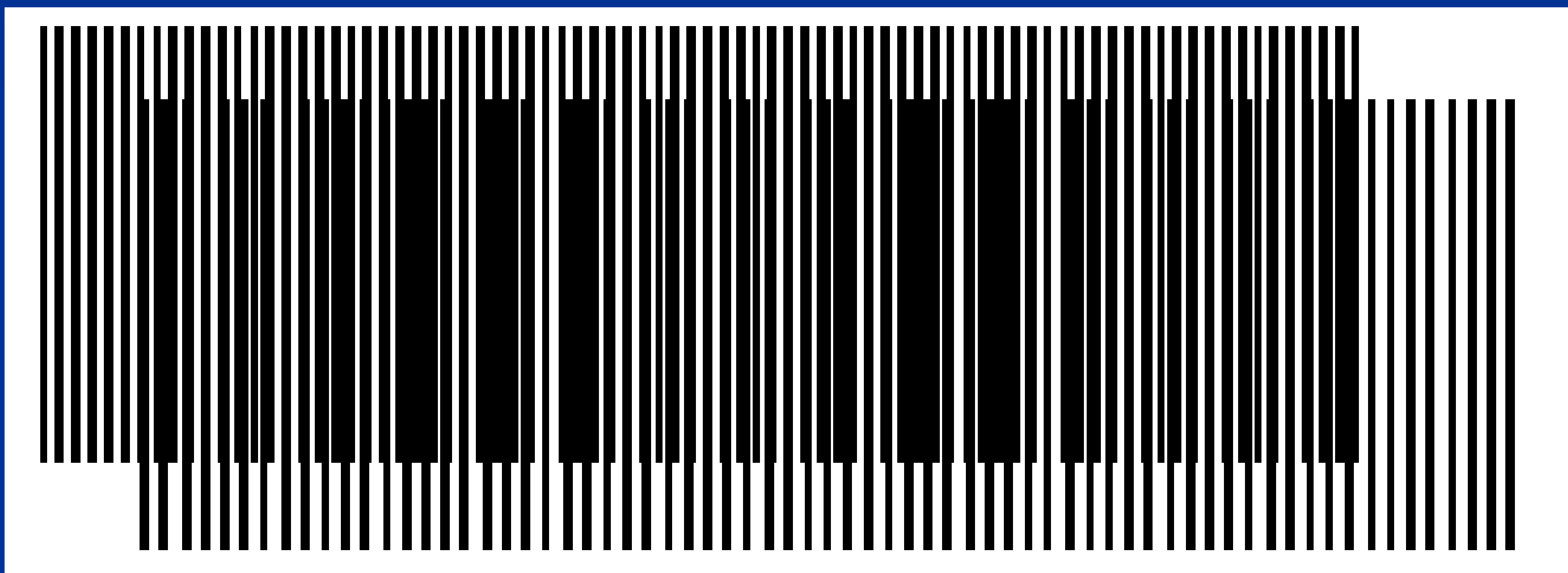


Loss of Detail



Jaggies

Moire Patterns



Moire Patterns